Bounds of Redefined Zagreb indices for F-Sum of Graphs, $F = Q$ or T [1]

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Abstract

Redefined Zagreb indices are new graph invariants, which is the degree based topological index $ReZG_1(G) = \sum_{uv\in E(G)} \frac{d_u+d_v}{d_u d_v}$, $ReZG_2(G) = \sum_{uv\in E(G)} \frac{d_u d_v}{d_u+d_v}$ and $ReZG_3(G) = \sum_{uv\in E(G)}(d_u d_v)$ $(d_u + d_v)$. Eliasi and Taeri introduced four new operations based on graphs *S(G), R(G), Q(G)* and $T(G)$, which are also known as *F-sum* of graphs, where *F= S, Q, R* and T. In this paper, we establish bounds of the redefined Zagreb indices for for F-Sum of Graphs, where $F = Q$ or T.

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Keywords and Phrases: Distance(in graphs), Redefined Zagreb indices, Operations on graphs, Subdivision of graph, Total graph.

2000 Mathematics Subject Classification: Primary 05C12; 05C90.

1 Basic definition, notation and preliminary results

All the graphs in this paper are simple, finite and undirected. In a graph G, $V(G)$ and $E(G)$ are the sets of vertices and edges respectively. Let $d_G(u)$ denotes the degree of a vertex u and $d_G(u,v)$ be the distance between two vertices u and v in G. Topological indices have been found to be useful in establishing relation between the structure and the properties of molecules. Topological indices mainly used in Quantitative Structure Property Relationship (QSPR) and Quantitative Structure Activity Relationships (QSAR)[1]. Some topological indices are degree based and some are distance based.

Wiener index [2, 3, 4] is a distance based topological index, denoted by *W(G)* and defined as the sum of distances over all unordered vertex pairs in G.

$$W(G) = \sum_{\{u,v\} \in V(G)} d_G(u,v)$$

The *Zagreb indices* were introduced more than thirty years ago by Gutman and Trinajstić [7]. After ten years, Balaban *et.al* named them *Zagreb group index*, presented by M_1 and M_2. Later it was abbreviated to *Zagreb index* [8], where M_1 and M_2 represents *first Zagreb index* and *second Zagreb index* respectively. If d_u and d_v are the degrees of vertices u, v for simple graph G. Then *first Zagreb index* [8, 9] is defined as

$$\begin{aligned} M_1(G) &= \sum_{v \in V(G)} (d_G(v))^2 \\ &= \sum_{uv \in E(G)} d_G(u) + d_G(v) \end{aligned}$$

Second Zagreb index is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

In 2013, Ranjini [16] introduced *redefined Zagreb indices i.e.*, *redefined first, second* and *third Zagreb indices* of a graph G. These are presented as

$$ReZG_1(G) = \sum_{uv \in E(G)} \frac{d_G(u) + d_G(v)}{d_G(u)d_G(v)}$$

$$ReZG_2(G) = \sum_{uv \in E(G)} \frac{d_G(u)d_G(v)}{d_G(u) + d_G(v)}$$

and

$$ReZG_3(G) = \sum_{uv \in E(G)} (d_G(u)d_G(v))(d_G(u) + d_G(v))$$

Since last thirty years, many scholars and researchers have been working on *composite graphs*. There are various graph operations which are applied directly on simple graphs to study their properties under these operations. Many authors computed several topological indices for these composite graphs [3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20], *e.g.* composition, disjunction, Cartesian product, corona product, indu-bala product and wreath product of two graphs.

Firstly we recall *Cartesian product* $G \Box K$, of graphs G and K. For the vertex set $V(G) \times V(K)$ in which $(p,q)(r,s)$ is an edge of $G \Box K$ if $[p = r$ and $qs \in E(K)]$ or $[q = s$ and $pr \in E(G)]$. If $d_{G\Box K}((p,q),(r,s))$ is the distance between any pair of vertices in $G \Box K$ then

$$d_{G\Box K}((p,q),(r,s)) = d_G(p,r) + d_K(q,s)$$

and degree of a vertex (p, q) of $G\Box K$

$$d_{G\Box K}(p, q) = d_G(p) + d_K(q)$$

In 2009, Eliasi *et.al* [4] used the notion of *F-sums* also known as *four graph operations*, which is actually the *Cartesian product* of $F(G_1)$ and G_2. *F-sum* of two graphs G_1 and G_2 is denoted by $G_1 +_F G_2$, where F be one of *S, R, Q, T* graph operation. These operations *i.e. S, R, Q, T* are defined as

1. Subdivision *S(G)* of a graph is acquired by embedding a vertex referred as the *white vertex* into each edge of *G*.

2. Two black vertices are *related* in $S(G)$ if they are adjacent in G. So $R(G)$ is obtained from $S(G)$ by joining each pair of *related* black vertices.

3. Similarly two white vertices are *related* in $S(G)$ if their corresponding edges are adjacent in G. $Q(G)$ is obtained by joining each pair of *related* white vertices.

4. Two graphs G and K having same vertex set V and edge set $E(G) \cup E(K)$ is called the union of G and K, denoted by $G \cup H$. In particular case total graph $T(G)$ is the union of $R(G)$ and $Q(G)$.

Many authors computed several topological indices for these four graph operations. Eliasi *et.al* [4] computed the wiener index of

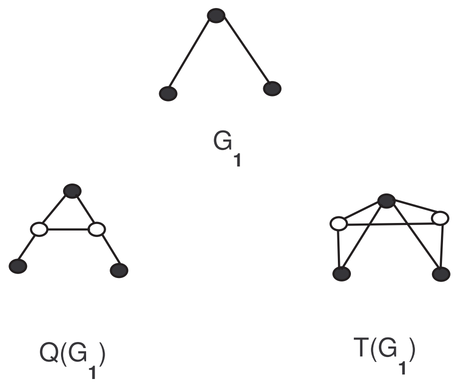

Figure 1: $F(G_1)$ for F= Q, T

these graph operations. In [5] Mingqiang An. *et.al* provided two upper bounds for the degree distance of F-sums of graphs. They used the results in [4] to find the distance between the vertices of F-sums of graphs. M. Imran *et.al* [12, 13] considered forgotten index and sum-connectivity index for these four graph operations and explored new results. They executed the exact value of forgotten index and sharp bounds of sum-connectivity index for each of these operations. M.H. Khalifeh *et.al* [8] worked on first and second Zagreb indices and they computed exact expressions for first and second Zagreb index for Cartesian product, composition, join, disjunction and symmetric difference of graphs. D. Sarala *et.al* [9] studied F-sums of graphs and find out exact formulas for first and second Zagreb indices. B. Basavangoud *et.al* [10] studied hyper-Zagreb coindex and hyper-Zagreb index for different graph operations. They calculated exact formula of hyper-Zagreb index for these four graph operations. Note that

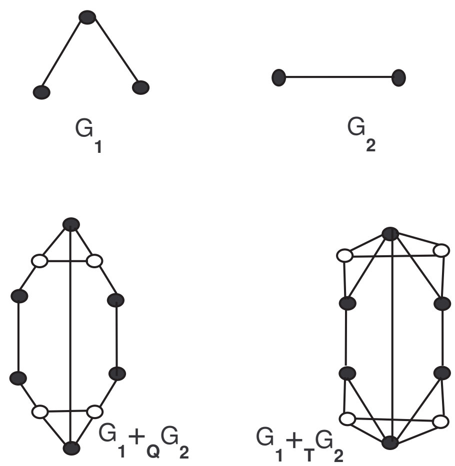

Figure 2: F-sums of G_1 and G_2 for F= Q, T

for *F-sum* of graph G_1 and G_2, take $|V_2| = n_2$ copies of the graph $F(G_1)$ and label them by the vertices of graph G_2. There are two situations for the vertices of $G_1 +_F G_2$: vertices V_1 referred to as black vertices and E_1 referred as white vertices . Now we join only black vertices with same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2.

Lemma 1.1. *Let G be a graph. Then:*
(a) *If $u_1 \in V(G)$, then we have*

$$d_{F(G)}(u_1) = k \cdot d_G(u_1),$$

where

$$k = \begin{cases} 1 & ;\ F\text{= } S \text{ or } Q \\ 2 & ;\ F\text{= } R \text{ or } T. \end{cases}$$

(b) *If* $u_1 = \acute{u}_1\acute{\acute{u}}_1 \in E(G)$*, then we have*

$$d_{S(G)}(u_1) = d_{R(G)}(u_1) = 2,$$

$$d_{Q(G)}(u_1) = d_{T(G)}(u_1) = d_{L(G)}(u_1) + 2,$$

where

$$d_{L(G)}(u_1) = d_G(\acute{u}_1) + d_G(\acute{\acute{u}}_1).$$

2 Redefined Zagreb indices for F-Sum of Graphs, when $F = Q$ or T

In this section, we established the bounds for *redefined Zagreb indices* in terms of Zagreb index for certain graph operations.

2.1 First Redefined Zagreb Index for F-sum of graphs, where $\boldsymbol{F}=\boldsymbol{Q}$ or $\boldsymbol{T}$

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First, we establish the results for the first redefined Zagreb index of $G_1 +_F G_2$ in terms of Δ_G maximum degree, δ_G minimum degree and Zagreb index of graph G.

Theorem 2.2. *Let* G_1 *and* G_2 *be two connected graphs with order* n_1, n_2*, size* m_1, m_2*, maximum degree* Δ_1*,* Δ_2 *and minimum degree* δ_1*,* δ_2 *respectively. Then first redefined Zagreb index for F-sum of graphs is*

$$\gamma_1 \leq ReZG_1(G_1 +_Q G_2) \leq \gamma_2$$

where

$$\gamma_1 = \frac{4m_1m_2 + n_1M_1(G_2)}{(\Delta_1+\Delta_2)^2} + \frac{4m_1m_2 + 3n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)} + \frac{4n_2\delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4(\Delta_1)^2}$$

and

$$\gamma_2 = \frac{4m_1m_2 + n_1M_1(G_2)}{(\delta_1+\delta_2^2)} + \frac{4m_1m_2 + 3n_2M_1(G_1)}{4\delta_1(\delta_1+\delta_2)} + \frac{4n_2\Delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4(\delta_1)^2}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of first redefined Zagreb index, we have

$$\begin{aligned} ReZG_1(G_1 +_Q G_2) &= \sum_{(u_i,v_k)(u_j,v_l)\in E(G_1+_QG_2)} \frac{d_{G_1+_QG_2}(u) + d_{G_1+_QG_2}(v)}{d_{G_1+_QG_2}(u)d_{G_1+_QG_2}(v)} \\ &= \sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_QG_2}(u_i,v_k) + d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)} \\ &+ \sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(Q(G_1))} \frac{d_{G_1+_QG_2}(u_i,v_k) + d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)} \end{aligned}$$

Note that $d_{G_1}(u) \leq \Delta_1$ and $d_{G_1}(u) \geq \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \leq \Delta_2$ and $d_{G_2}(v) \geq \delta_2$, equality holds if and only if G_2 is a regular graph. We get

$$\begin{aligned} &\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_QG_2}(u_i,v_k) + d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)} \\ &= \sum_{u\in V_1}\sum_{v_kv_l\in E_2} \frac{[d_{Q(G_1)}(u) + d_{G_2}(v_k)] + [d_{Q(G_1)}(u) + d_{G_2}(v_l)]}{[d_{Q(G_1)}(u) + d_{G_2}(v_k)][d_{Q(G_1)}(u) + d_{G_2}(v_l)]} \\ &\geq \frac{1}{(\Delta_1+\Delta_2)^2}\Bigg(\sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_1}(u) + d_{G_1}(u)) + \sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_2}(v_k) + d_{G_2}(v_l))\Bigg) \\ &\geq \frac{1}{(\Delta_1+\Delta_2)^2}\Big(m_2(4m_1) + n_1M_1(G_2)\Big) \\ &\geq \frac{4m_1m_2 + n_1M_1(G_2)}{(\Delta_1+\Delta_2)^2} \end{aligned}$$

Since $|E(Q(G))| = 2|E(G)|$ and $\Delta_{Q(G)} = \Delta_G$

$$\sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(Q(G_1))}\frac{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}$$

Now,

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}\frac{d_{Q(G_1)}(u_i)+d_{G_2}(v_k)+d_{Q(G_1)}(u_j)}{d_{Q(G_1)}(u_i)+d_{G_2}(v_k)(d_{Q(G_1)}(u_j))}$$
$$\geq\frac{1}{4\Delta_1(\Delta_1+\Delta_2)}\Bigg(\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}(d_{Q(G_1)}(u_j))$$
$$+n_2M_1(G_1)+4m_1m_2\Bigg)$$
$$\geq\frac{4m_1m_2+n_2M_1(G_1)+2n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)}$$
$$\geq\frac{4m_1m_2+3n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)}$$

and

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}\frac{d_{Q(G_1)}(u_i)+d_{Q(G_1)}(u_j)}{(d_{Q(G_1)}(u_i))(d_{Q(G_1)}(u_j))}$$

where u_i is the common vertex for w_1 and w_2 in $Q(G_1)$, $e = w_1w_2 \in E(G_1)$. Seems like, u_i inserted in edge w_1w_2 and u_j inserted in edge w_2w_3 of G_1. So we have,

$$=\sum_{v\in V_2}\sum_{w_1w_2\in E(G),w_2w_3\in E(G_1)}\frac{d_{G_1}(w_1)+d_{G_1}(w_2)+d_{G_1}(w_2)+d_{G_1}(w_3)}{[d_{G_1}(w_1)+d_{G_1}(w_2)][d_{G_1}(w_2)+d_{G_1}(w_3)]}$$
$$\geq\frac{4n_2\delta_1(\frac{1}{2}M_1(G_1)-m_1)}{4(\Delta_1)^2}$$

By using these values, we get the required equation

$$ReZG_1(G_{1+_Q G_2}) \geq \frac{4m_1m_2 + n_1M_1(G_2)}{(\Delta_1+\Delta_2)^2} + \frac{4m_1m_2 + 3n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)} + \frac{4n_2\delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4(\Delta_1)^2}.$$

Similarly we can compute

$$ReZG_1(G_{1+_Q G_2}) \leq \frac{4m_1m_2 + n_1M_1(G_2)}{(\delta_1+\delta_2)^2} + \frac{4m_1m_2 + 3n_2M_1(G_1)}{4\delta_1(\delta_1+\delta_2)} + \frac{4n_2\Delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4(\delta_1)^2}$$

□

Theorem 2.3. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2, maximum degree Δ_1, Δ_2 and minimum degree δ_1, δ_2 respectively. Then first redefined Zagreb index for F-sum of graphs is*

$$\gamma_1 \leq ReZG_1(G_1 +_T G_2) \leq \gamma_2$$

where

$$\gamma_1 = \frac{4m_1m_2 + 4n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)} + \frac{4n_2\delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4{\Delta_1}^2} + \frac{12m_1m_2 + n_1M_1(G_2) + n_2M_1(G_1)}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}$$

and

$$\gamma_2 = \frac{4m_1m_2 + 4n_2M_1(G_1)}{4\delta_1(\delta_1+\delta_2)} + \frac{4n_2\Delta_1(\frac{1}{2}M_1(G_1) - m_1)}{4{\delta_1}^2} + \frac{12m_1m_2 + n_1M_1(G_2) + n_2M_1(G_1}{2\delta_1\delta_2(2\delta_1+\delta_2)}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of first redefined Zagreb index, we have

$$
\begin{aligned}
ReZG_1(G_1 +_T G_2) &= \sum_{(u_i,v_k)(u_j,v_l)\in E(G_1+_T G_2)} \frac{d_{G_1+_TG_2}(u) + d_{G_1+_TG_2}(v)}{d_{G_1+_TG_2}(u)d_{G_1+_TG_2}(v)} \\
&= \sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)} \\
&+ \sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(T(G_1))} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}
\end{aligned}
$$

Note that $d_{G_1}(u) \le \Delta_1$ and $d_{G_1}(u) \ge \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \le \Delta_2$ and $d_{G_2}(v) \ge \delta_2$, equality holds if and only if G_2 is a regular graph. We get

$$
\begin{aligned}
&\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)} \\
&= \sum_{u\in V_1}\sum_{v_kv_l\in E_2} \frac{[d_{T(G_1)}(u) + d_{G_2}(v_k)] + [d_{T(G_1)}(u) + d_{G_2}(v_l)]}{[d_{T(G_1)}(u) + d_{G_2}(v_k)][d_{T(G_1)}(u) + d_{G_2}(v_l)]} \\
&= \sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{[2d_{G_1}(u) + d_{G_2}(v_k)] + [2d_{G_1}(u) + d_{G_2}(v_l)]}{[2d_{G_1}(u) + d_{G_2}(v_k)][2d_{G_1}(u) + d_{G_2}(v_l)]} \\
&\ge \frac{1}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}\Bigg(2\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2}(d_{G_1}(u) + d_{G_1}(u)) \\
&+ \sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_2}(v_k) + d_{G_2}(v_l))\Bigg) \\
&\ge \frac{1}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}\Big(4(m_2)(2m_1) + n_1M_1(G_2)\Big) \\
&\ge \frac{8m_1m_2 + n_1M_1(G_2)}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}
\end{aligned}
$$

Since $|E(T(G))| = 2|E(G)|$ and $\Delta_{T(G)} = 2\Delta_G$

$$
\begin{aligned}
&\sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(T(G_1))} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)} \\
&= \sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)} \\
&+ \sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i\in V(G_1),u_j\in V(T(G_1))-V(G_1)} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)} \\
&+ \sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(T(G_1))-V(G_1)} \frac{d_{G_1+_TG_2}(u_i,v_k) + d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}
\end{aligned}
$$

Note that $u_i, u_j \in V(G_1)$ and $u_iu_j \in E(R(G_1))$ if and only if $u_iu_j \in E(G_1)$, we have

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)} \frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$
$$\geq \frac{1}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}\Big(\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(d_{T(G_1)}(u_i)+d_{G_2}(v))$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(d_{T(G_1)}(u_j)+d_{G_2}(v))\Big)$$
$$\geq \frac{1}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}\Big(\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(2d_{G_1}(u_i)+2d_{G_1}(u_j))$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(d_{G_2}(v)+d_{G_2}(v))\Big)$$
$$\geq \frac{1}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}\Big(n_2(2M_1(G_1))+2m_1(2m_2)\Big)$$
$$\geq \frac{4m_1m_2+2n_2M_1(G_1)}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}$$

Since $d_{G_1+_TG_2}(u,v) = d_{G_1+_QG_2}(u,v)$ for $u \in V(T(G_1) - V(G_1))$ and $v \in V(G_2)$, we get the following equation by using the proof of Theorem2.2, so we have required equation as

$$ReZG_1(G_1+_T G_2) \geq \frac{4m_1m_2+4n_2M_1(G_1)}{4\Delta_1(\Delta_1+\Delta_2)} + \frac{4n_2\delta_1(\frac{1}{2}M_1(G_1)-m_1)}{4{\Delta_1}^2}$$
$$+\frac{12m_1m_2+n_1M_1(G_2)+n_2M_1(G_1)}{2\Delta_1\Delta_2(2\Delta_1+\Delta_2)}$$

and

$$ReZG_1(G_1+_T G_2) \leq \frac{4m_1m_2+4n_2M_1(G_1)}{4\delta_1(\delta_1+\delta_2)} + \frac{4n_2\Delta_1(\frac{1}{2}M_1(G_1)-m_1)}{4{\delta_1}^2}$$
$$+\frac{12m_1m_2+n_1M_1(G_2)+n_2M_1(G_1}{2\delta_1\delta_2(2\delta_1+\delta_2)}$$

□

2.4 Second Redefined Zagreb Index for F-sum of graphs, where $F = Q$ or T

.

In this section, we establish the results for the second redefined Zagreb index of $G_1 +_F G_2$ in terms of Δ_G maximum degree, δ_G minimum degree and Zagreb index of graph G.

Theorem 2.5. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2, maximum degree Δ_1, Δ_2 and minimum degree δ_1, δ_2 respectively. Then second redefined Zagreb index for F-sum of graphs is*

$$\gamma_1 \leq ReZG_2(G_1 +_Q G_2) \leq \gamma_2 \tag{1}$$

where

$$\begin{aligned}
\gamma_1 = \frac{m_2 M_1(G_1) + 2m_1 M_1(G_2) + n_1 M_2(G_2)}{2(\Delta_1 + \Delta_2)} &+ \frac{4m_2 M_1(G_1) + 8m_1 n_2 \delta_1^2}{4\Delta_1(\Delta_1 + \Delta_2)} \\
&+ \frac{4n_2 {\delta_1}^2(\frac{1}{2} M_1(G_1) - m_1)}{4\Delta_1} \\
\gamma_1 = \frac{m_2 M_1(G_1) + 2m_1 M_1(G_2) + n_1 M_2(G_2)}{2(\delta_1 + \delta_2)} &+ \frac{4m_2 M_1(G_1) + 8m_1 n_2 \Delta_1^2}{4\delta_1(\delta_1 + \delta_2)} \\
&+ \frac{4n_2 {\Delta_1}^2(\frac{1}{2} M_1(G_1) - m_1)}{4\delta_1}
\end{aligned} \tag{2}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of second redefined Zagreb index we have

$$ReZG_2(G_1 +_Q G_2) = \sum_{(u_i,v_k)(u_j,v_l)\in E(G_1+_QG_2)} \frac{d_{G_1+_QG_2}(u)d_{G_1+_QG_2}(v)}{d_{G_1+_QG_2}(u)+d_{G_1+_QG_2}(v)}$$
$$= \sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$+ \sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(Q(G_1))} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$

Note that $d_{G_1}(u) \leq \Delta_1$ and $d_{G_1}(u) \geq \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \leq \Delta_2$ and $d_{G_2}(v) \geq \delta_2$, equality holds if and only if G_2 is a regular graph. We get

$$\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$= \sum_{u\in V_1}\sum_{v_kv_l\in E_2} \frac{[d_{Q(G_1)}(u)+d_{G_2}(v_k)][d_{Q(G_1)}(u)+d_{G_2}(v_l)]}{[d_{Q(G_1)}(u)+d_{G_2}(v_k)]+[d_{Q(G_1)}(u)+d_{G_2}(v_l)]}$$
$$\geq \frac{1}{2(\Delta_1+\Delta_2)}\Big(\sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_1}(u)+d_{G_2}(v_k))(d_{G_1}(u)+d_{G_2}(v_l))\Big)$$
$$\geq \frac{1}{2(\Delta_1+\Delta_2)}\Big(m_2M_1(G_1)+2m_1M_1(G_2)+n_1M_2(G_2)\Big)$$
$$\geq \frac{m_2M_1(G_1)+2m_1M_1(G_2)+n_1M_2(G_2)}{2(\Delta_1+\Delta_2)}$$

Since $|E(Q(G))| = 2|E(G)|$ and $\Delta_{Q(G)} = \Delta_G$

$$\sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(Q(G_1))} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$= \sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$+ \sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)} \frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$

Now,

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}\frac{(d_{Q(G_1)}(u_i)+d_{G_2}(v_k))(d_{Q(G_1)}(u_j))}{(d_{Q(G_1)}(u_i)+d_{G_2}(v_k))+(d_{Q(G_1)}(u_j))}$$
$$\geq\frac{1}{4\Delta_1(\Delta_1+\Delta_2)}\Big(\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}d_{G_1}(u_i)(d_{Q(G_1)}(u_j))$$
$$+4m_2M_1(G_1)\Big)$$
$$\geq\frac{4m_2M_1(G_1)+4n_2\delta_1^2(2m_1)}{4\Delta_1(\Delta_1+\Delta_2)}$$
$$\geq\frac{4m_2M_1(G_1)+8m_1n_2\delta_1^2}{4\Delta_1(\Delta_1+\Delta_2)}$$

and

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}\frac{d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)}{d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}\frac{d_{Q(G_1)}(u_i)d_{Q(G_1)}(u_j)}{(d_{Q(G_1)}(u_i))+(d_{Q(G_1)}(u_j))}$$

where u_i inserted in edge w_1w_2 and u_j inserted in edge w_2w_3 of G_1. So we have,

$$=\sum_{v\in V_2}\sum_{w_1w_2\in E(G),w_2w_3\in E(G_1)}\frac{[d_{G_1}(w_1)+d_{G_1}(w_2)][d_{G_1}(w_2)+d_{G_1}(w_3)]}{d_{G_1}(w_1)+d_{G_1}(w_2)+d_{G_1}(w_2)+d_{G_1}(w_3)}$$
$$\geq\frac{4n_2{\delta_1}^2(\frac{1}{2}M_1(G_1)-m_1)}{4\Delta_1}$$

By using these values, we get the required equation

$$ReZG_2(G_1+_QG_2)\geq\frac{m_2M_1(G_1)+2m_1M_1(G_2)+n_1M_2(G_2)}{2(\Delta_1+\Delta_2)}+\frac{4m_2M_1(G_1)+8m_1n_2\delta_1^2}{4\Delta_1(\Delta_1+\Delta_2)}$$
$$+\frac{4n_2{\delta_1}^2(\frac{1}{2}M_1(G_1)-m_1)}{4\Delta_1}.$$

Similarly we can compute

$$ReZG_2(G_1+_SG_2)\leq\frac{m_2M_1(G_1)+2m_1M_1(G_2)+n_1M_2(G_2)}{2(\delta_1+\delta_2)}+\frac{4m_2M_1(G_1)+8m_1n_2\Delta_1^2}{4\delta_1(\delta_1+\delta_2)}$$
$$+\frac{4n_2{\Delta_1}^2(\frac{1}{2}M_1(G_1)-m_1)}{4\delta_1}$$

□

Theorem 2.6. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2, maximum degree Δ_1, Δ_2 and minimum degree δ_1, δ_2 respectively. Then second redefined Zagreb index for F-sum of graphs is*

$$\gamma_1 \leq ReZG_2(G_1 +_T G_2) \leq \gamma_2$$

where

$$\gamma_1 = \frac{4m_2M_1(G_1) + 16m_1n_2\delta_1^2}{4\Delta_1(\Delta_1 + \Delta_2)} + \frac{4n_2\delta_1^2(\frac{1}{2}M_1(G_1) - m_1)}{4\Delta_1}$$
$$\frac{8m_2M_1(G_1) + 5m_1M_1(G_2) + n_1M_2(G_2) + 4n_2M_2(G_1)}{2(2\Delta_1 + \Delta_2)}$$

and

$$\gamma_2 = \frac{4m_2M_1(G_1) + 16m_1n_2\Delta_1^2}{4\delta_1(\delta_1 + \delta_2)} + \frac{4n_2\Delta_1^2(\frac{1}{2}M_1(G_1) - m_1)}{4\delta_1}$$
$$\frac{8m_2M_1(G_1) + 5m_1M_1(G_2) + n_1M_2(G_2) + 4n_2M_2(G_1)}{2(2\Delta_1 + \Delta_2)}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of second redefined Zagreb index, we have

$$\begin{aligned} ReZG_2(G_1 +_T G_2) &= \sum_{(u_i,v_k)(u_j,v_l) \in E(G_1 +_T G_2)} \frac{d_{G_1+_TG_2}(u) d_{G_1+_TG_2}(v)}{d_{G_1+_TG_2}(u) + d_{G_1+_TG_2}(v)} \\ &= \sum_{u_i = u_j \in V_1} \sum_{v_k v_l \in E_2} \frac{d_{G_1+_TG_2}(u_i, v_k) d_{G_1+_TG_2}(u_j, v_l)}{d_{G_1+_TG_2}(u_i, v_k) + d_{G_1+_TG_2}(u_j, v_l)} \\ &+ \sum_{v_k = v_l \in V_2} \sum_{u_i u_j \in E(T(G_1))} \frac{d_{G_1+_TG_2}(u_i, v_k) d_{G_1+_TG_2}(u_j, v_l)}{d_{G_1+_TG_2}(u_i, v_k) + d_{G_1+_TG_2}(u_j, v_l)} \end{aligned}$$

Note that $d_{G_1}(u) \leq \Delta_1$ and $d_{G_1}(u) \geq \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \leq \Delta_2$ and $d_{G_2}(v) \geq \delta_2$, equality holds if and only if G_2 is a regular graph.

We get

$$\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2}\frac{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}$$
$$=\sum_{u\in V_1}\sum_{v_kv_l\in E_2}\frac{[d_{T(G_1)}(u)+d_{G_2}(v_k)][d_{T(G_1)}(u)+d_{G_2}(v_l)]}{[d_{T(G_1)}(u)+d_{G_2}(v_k)]+[d_{T(G_1)}(u)+d_{G_2}(v_l)]}$$
$$=\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2}\frac{[2d_{G_1}(u)+d_{G_2}(v_k)][2d_{G_1}(u)+d_{G_2}(v_l)]}{[2d_{G_1}(u)+d_{G_2}(v_k)]+[2d_{G_1}(u)+d_{G_2}(v_l)]}$$
$$\geq\frac{1}{2(2\Delta_1+\Delta_2)}\Bigg(4\sum_{u\in V_1}\sum_{v_kv_l\in E_2}d^2_{G_1}(u)+2\sum_{u\in V_1}\sum_{v_kv_l\in E_2}d_{G_1}(u)(d_{G_2}(v_k)+d_{G_2}(v_l))$$
$$+\sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_2}(v_k)d_{G_2}(v_l))\Bigg)$$
$$\geq\frac{4m_2M_1(G_1)+4m_1M_1(G_2)+n_1M_2(G_2)}{2(2\Delta_1+\Delta_2)}$$

Since $|E(T(G))|=2|E(G)|$ and $\Delta_{T(G)}=2\Delta_G$

$$\sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(T(G_1))}\frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$
$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}\frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$

$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i\in V(G_1),u_j\in V(T(G_1))-V(G_1)}\frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(T(G_1))-V(G_1)}\frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(T(G_1))-V(G_1)}\frac{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}$$

Note that $u_i,u_j\in V(G_1)$ and $u_iu_j\in E(T(G_1))$ if and only if

$u_iu_j \in E(G_1)$, we have

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)} \frac{d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)}{d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)}$$
$$\geq \frac{1}{2(2\Delta_1+\Delta_2)}\Big(\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(d_{T(G_1)}(u_i)$$
$$+d_{G_2}(v))(d_{T(G_1)}(u_j)+d_{G_2}(v))\Big)$$
$$\geq \frac{1}{2(2\Delta_1+\Delta_2)}\Big(2\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(d_{G_2}(v))(d_{G_1}(u_i)+d_{G_1}(u_j))$$
$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}(4d_{G_1}(u_i)d_{G_1}(u_j)+d^2_{G_2}(v))\Big)$$
$$\geq \frac{1}{2(2\Delta_1+\Delta_2)}\Big(4m_2M_1(G_1)+n_2(4M_2(G_1))+m_1(M_1(G_2))\Big)$$
$$\geq \frac{4m_2M_1(G_1)+4n_2M_2(G_1)+m_1M_1(G_2)}{2(2\Delta_1+\Delta_2)}$$

Since $d_{G_1+_TG_2}(u,v) = d_{G_1+_QG_2}(u,v)$ for $u \in V(T(G_1) - V(G_1))$ and $v \in V(G_2)$, we get the following equation by using the proof of Theorem2.5

$$ReZG_2(G_1+_TG_2) \geq \frac{4m_2M_1(G_1)+16m_1n_2\delta_1^2}{4\Delta_1(\Delta_1+\Delta_2)} + \frac{4n_2\delta_1^2(\frac{1}{2}M_1(G_1)-m_1)}{4\Delta_1}$$
$$\frac{8m_2M_1(G_1)+5m_1M_1(G_2)+n_1M_2(G_2)+4n_2M_2(G_1)}{2(2\Delta_1+\Delta_2)}$$

and

$$ReZG_2(G_1+_TG_2) \leq \frac{4m_2M_1(G_1)+16m_1n_2\Delta_1^2}{4\delta_1(\delta_1+\delta_2)} + \frac{4n_2\Delta_1^2(\frac{1}{2}M_1(G_1)-m_1)}{4\delta_1}$$
$$\frac{8m_2M_1(G_1)+5m_1M_1(G_2)+n_1M_2(G_2)+4n_2M_2(G_1)}{2(2\Delta_1+\Delta_2)}$$

□

2.7 Third Redefined Zagreb Index for F-sum of graphs, where *F*= *Q* or *T*

.

In this section, we establish the results for the third redefined

Zagreb index of $G_1 +_F G_2$ in terms of Δ_G maximum degree, δ_G minimum degree and Zagreb index of graph G.

Theorem 2.8. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2, maximum degree Δ_1, Δ_2 and minimum degree δ_1, δ_2 respectively. Then third redefined Zagreb index for F-sum of graphs is*

$$\gamma_1 \leq ReZG_3(G_1 +_Q G_2) \leq \gamma_2$$

where

$$\begin{aligned}
\gamma_1 =& 2(\delta_1+\delta_2)^3 + 8m_1n_2\delta_1(\delta_1+\delta_2)(5\delta_1+\delta_2) + 16n_2\delta_1^3(\frac{1}{2}M_1(G_1) - m_1) \\
\gamma_2 =& 2(\Delta_1+\Delta_2)^3 + 8m_1n_2\Delta_1(\Delta_1+\Delta_2)(5\Delta_1+\Delta_2) + 16n_2\Delta_1^3(\frac{1}{2}M_1(G_1) - m_1)
\end{aligned} \tag{3}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of third redefined Zagreb index we have

$$\begin{aligned}
ReZG_3(G_1 +_Q G_2) =& \sum_{(u_i,v_k)(u_j,v_l)\in E(G_1+_QG_2)} [d_{G_1+_QG_2}(u)d_{G_1+_QG_2}(v)][d_{G_1+_QG_2}(u) \\
&+ d_{G_1+_QG_2}(v)] \\
&= \sum_{u_i=u_j\in V_1} \sum_{v_kv_l\in E_2} [d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)] \\
&[d_{G_1+_QG_2}(u_i,v_k) + d_{G_1+_QG_2}(u_j,v_l)] \\
&+ \sum_{v_k=v_l\in V_2} \sum_{u_iu_j\in E(Q(G_1))} [d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)] \\
&[d_{G_1+_QG_2}(u_i,v_k) + d_{G_1+_QG_2}(u_j,v_l)]
\end{aligned}$$

Note that $d_{G_1}(u) \leq \Delta_1$ and $d_{G_1}(u) \geq \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \leq \Delta_2$ and $d_{G_2}(v) \geq \delta_2$, equality holds if and only if G_2 is a regular graph.

We get

$$\sum_{u_i=u_j\in V_1}\sum_{v_kv_l\in E_2}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)][d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)]$$

$$=\sum_{u\in V_1}\sum_{v_kv_l\in E_2}[(d_{Q(G_1)}(u)+d_{G_2}(v_k))(d_{Q(G_1)}(u)+d_{G_2}(v_l))]$$

$$[(d_{Q(G_1)}(u)+d_{G_2}(v_k))+(d_{Q(G_1)}(u)+d_{G_2}(v_l))]$$

$$\leq 2(\Delta_1+\Delta_2)\Big(\sum_{u\in V_1}\sum_{v_kv_l\in E_2}(d_{G_1}(u)+d_{G_2}(v_k))(d_{G_1}(u)+d_{G_2}(v_l))\Big)$$

$$\leq 2n_1m_2(\Delta_1+\Delta_2)(\Delta_1+\Delta_2)^2$$

$$\leq 2(\Delta_1+\Delta_2)^3$$

Since $|E(Q(G))|=2|E(G)|$ and $\Delta_{Q(G)}=\Delta_G$

$$\sum_{v_k=v_l\in V_2}\sum_{u_iu_j\in E(Q(G_1))}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)][d_{G_1+_QG_2}(u_i,v_k)$$

$$+d_{G_1+_QG_2}(u_j,v_l)]$$

$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)]$$

$$[d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)]$$

$$+\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)]$$

$$[d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)]$$

Now,

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)]$$

$$[d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)]$$

$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i\in V(G_1),u_j\in V(Q(G_1))(G_1)}[(d_{Q(G_1)}(u_i)+d_{G_2}(v_k))(d_{Q(G_1)}(u_j))]$$

$$[(d_{Q(G_1)}(u_i)+d_{G_2}(v_k))+(d_{Q(G_1)}(u_j))]$$

$$\leq [4n_2\Delta_1(2m_1)(\Delta_1+\Delta_2)][(\Delta_1+\Delta_2)+4\Delta_1]$$

$$\leq [4n_2\Delta_1(2m_1)(\Delta_1+\Delta_2)][(5\Delta_1+\Delta_2)]$$

$$\leq 8m_1n_2\Delta_1(\Delta_1+\Delta_2)(5\Delta_1+\Delta_2)$$

and

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}[d_{G_1+_QG_2}(u_i,v_k)d_{G_1+_QG_2}(u_j,v_l)]$$

$$[d_{G_1+_QG_2}(u_i,v_k)+d_{G_1+_QG_2}(u_j,v_l)]$$

$$=\sum_{v\in V_2}\sum_{u_iu_j\in E(Q(G_1));u_i,u_j\in V(Q(G_1))(G_1)}[d_{Q(G_1)}(u_i)d_{Q(G_1)}(u_j)][(d_{Q(G_1)}(u_i))$$

$$+(d_{Q(G_1)}(u_j))]$$

Where u_i inserted in edge w_1w_2 and u_j inserted in edge w_2w_3 of G_1. So we have,

$$= \sum_{v\in V_2} \sum_{w_1w_2\in E(G), w_2w_3\in E(G_1)} [(d_{G_1}(w_1) + d_{G_1}(w_2))(d_{G_1}(w_2) + d_{G_1}(w_3))]$$
$$[d_{G_1}(w_1) + d_{G_1}(w_2) + d_{G_1}(w_2) + d_{G_1}(w_3)]$$
$$\leq 16n_2\Delta_1^3(\frac{1}{2}M_1(G_1) - m_1)$$

By using these values, we get the required equation

$$\begin{aligned} ReZG_3(G_{1+_QG_2}) \leq & 2(\Delta_1 + \Delta_2)^3 + 8m_1n_2\Delta_1(\Delta_1 + \Delta_2)(5\Delta_1 + \Delta_2) \\ & + 16n_2\Delta_1^3(\frac{1}{2}M_1(G_1) - m_1). \end{aligned}$$

Similarly we can compute

$$ReZG_3(G_{1+_QG_2}) \geq 2(\delta_1 + \delta_2)^3 + 8m_1n_2\delta_1(\delta_1 + \delta_2)(5\delta_1 + \delta_2) + 16n_2\delta_1^3(\frac{1}{2}M_1(G_1) - m_1)$$

□

Theorem 2.9. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2, maximum degree Δ_1, Δ_2 and minimum degree δ_1, δ_2 respectively. Then third redefined Zagreb index for F-sum of graphs G is*

$$\gamma_1 \leq ReZG_3(G_1 +_T G_2) \leq \gamma_2$$

where

$$\begin{aligned} \gamma_1 = & 2(m_1n_2 + n_1m_2)(2\Delta_1 + \Delta_2)^3 + 16n_2\Delta_1^3(\frac{1}{2}M_1(G_1) - m_1) \\ & + 8m_1n_2\Delta_1(2\Delta_1 + \Delta_2)(6\Delta_1 + \Delta_2) \end{aligned}$$

and

$$\begin{aligned} \gamma_2 = & 2(m_1n_2 + n_1m_2)(2\delta_1 + \delta_2)^3 + 16n_2\delta_1^3(\frac{1}{2}M_1(G_1) - m_1) \\ & + 8m_1n_2\delta_1(2\delta_1 + \delta_2)(6\delta_1 + \delta_2) \end{aligned}$$

equality holds if and only if G_2 is a regular graph.

Proof. By the definition of third redefined Zagreb index, we have

$$ReZG_3(G_1 +_T G_2) = \sum_{(u_i,v_k)(u_j,v_l)\in E(G_1+_T G_2)} [d_{G_1+_T G_2}(u) + d_{G_1+_T G_2}(v)]$$
$$[d_{G_1+_T G_2}(u) d_{G_1+_T G_2}(v)]$$
$$= \sum_{u_i=u_j\in V_1} \sum_{v_k v_l\in E_2} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$
$$+ \sum_{v_k=v_l\in V_2} \sum_{u_i u_j\in E(T(G_1))} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$

Note that $d_{G_1}(u) \leq \Delta_1$ and $d_{G_1}(u) \geq \delta_1$, equality holds if and only if G_1 is a regular graph, and similarly $d_{G_2}(v) \leq \Delta_2$ and $d_{G_2}(v) \geq \delta_2$, equality holds if and only if G_2 is a regular graph. We get

$$\sum_{u_i=u_j\in V_1} \sum_{v_k v_l\in E_2} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$
$$= \sum_{u\in V_1} \sum_{v_k v_l\in E_2} [(d_{T(G_1)}(u) + d_{G_2}(v_k)) + (d_{T(G_1)}(u) + d_{G_2}(v_l))]$$
$$[(d_{T(G_1)}(u) + d_{G_2}(v_k))(d_{T(G_1)}(u) + d_{G_2}(v_l))]$$
$$= \sum_{u_i=u_j\in V_1} \sum_{v_k v_l\in E_2} [(2d_{G_1}(u) + d_{G_2}(v_k)) + (2d_{G_1}(u) + d_{G_2}(v_l))]$$
$$[(2d_{G_1}(u) + d_{G_2}(v_k))(2d_{G_1}(u) + d_{G_2}(v_l))]$$
$$\leq n_1 m_2 \Big(2(2\Delta_1 + \Delta_2)(2\Delta_1 + \Delta_2)^2 \Big)$$
$$\leq 2 n_1 m_2 (2\Delta_1 + \Delta_2)^3$$

Since $|E(T(G))| = 2|E(G)|$ and $\Delta_{T(G)} = 2\Delta_G$

$$\sum_{v_k=v_l\in V_2} \sum_{u_i u_j\in E(T(G_1))} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$
$$= \sum_{v\in V_2} \sum_{u_i u_j\in E(T(G_1));u_i,u_j\in V(G_1)} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$
$$+ \sum_{v\in V_2} \sum_{u_i u_j\in E(T(G_1));u_i\in V(G_1),u_j\in V(T(G_1))-V(G_1)} [d_{G_1+_T G_2}(u_i,v_k) + d_{G_1+_T G_2}(u_j,v_l)]$$
$$[d_{G_1+_T G_2}(u_i,v_k) d_{G_1+_T G_2}(u_j,v_l)]$$

Note that $u_i, u_j \in V(G_1)$ and $u_iu_j \in E(T(G_1))$ if and only if $u_iu_j \in E(G_1)$, we have

$$\sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}[d_{G_1+_TG_2}(u_i,v_k)+d_{G_1+_TG_2}(u_j,v_l)]$$
$$[d_{G_1+_TG_2}(u_i,v_k)d_{G_1+_TG_2}(u_j,v_l)]$$
$$\geq \sum_{v\in V_2}\sum_{u_iu_j\in E(T(G_1));u_i,u_j\in V(G_1)}[(d_{T(G_1)}(u_i)+d_{G_2}(v))+(d_{T(G_1)}(u_j)+d_{G_2}(v))]$$
$$[(d_{T(G_1)}(u_i)+d_{G_2}(v))(d_{T(G_1)}(u_j)+d_{G_2}(v))]$$
$$\geq 2m_1n_2(2\Delta_1+\Delta_2)^3$$

Since $d_{G_1+_TG_2}(u,v) = d_{G_1+_QG_2}(u,v)$ for $u \in V(T(G_1)-V(G_1))$ and $v \in V(G_2)$, we get the following equation by using the proof of Theorem2.8

$$\begin{aligned}ReZG_2(G_1+_T G_2) \leq & 2(m_1n_2+n_1m_2)(2\Delta_1+\Delta_2)^3+16n_2\Delta_1^3(\frac{1}{2}M_1(G_1)-m_1)\\ &+8m_1n_2\Delta_1(2\Delta_1+\Delta_2)(6\Delta_1+\Delta_2)\end{aligned}$$

and

$$\begin{aligned}ReZG_2(G_1+_T G_2) \geq & 2(m_1n_2+n_1m_2)(2\delta_1+\delta_2)^3+16n_2\delta_1^3(\frac{1}{2}M_1(G_1)-m_1)\\ &+8m_1n_2\delta_1(2\delta_1+\delta_2)(6\delta_1+\delta_2)\end{aligned}$$

□

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