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Some properties of solitary BGK waves in a collisionless plasma

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Following the method developed by Bernstein, Greene & Kruskal we obtain expressions for the distribution function of the trapped particles for a collisionless plasma, for cases when the plasma is unbounded and bounded (in a cylindrical waveguide). Figures are drawn showing the relationship between the width and amplitude of the solitary BGK wave for various cases. Analytic expressions depicting this relationship are also derived.

0. Introduction

In experiments on Q-machines, Lynov *et al.* (1979) observed the formation and consequent motion of solitary Bernstein–Greene–Kruskal (BGK) waves. Similar results were obtained by Turikov (1978), in computer simulation experiments.

In the present work, we follow Berstein, Green & Kruskal (1957) in obtaining expressions for the unknown distribution function of the trapped particles. It is assumed that the distribution function of the untrapped particles and the electrostatic potential profile are known. The electrostatic potential is chosen in the form of a standard soliton solution and we investigate the relationship between the amplitude and width of the solitary BGK wave with the condition that the distribution function of the trapped particles is non-negative. Propagation is considered in one dimension only.

The paper is arranged in the following manner. § 1 is devoted to the mathematical formulation of the problem. The plasma is taken to be unbounded and unmagnetized. General expressions are obtained for the distribution function of the trapped particles. In § 2 expressions are obtained for the distribution function of the trapped particles for two types of distribution function (step-Type and Maxwellian) of the untrapped particles. We then derive expressions for δ (the width) and ψ (the amplitude) of the solitary BGK wave. In § 3 we proceed with an

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analysis of the dependance of δ upon ψ. In § 4 we consider the propagation of solitary BGK waves when the plasma is magnetized and situated in a cylindrical wavequide. An expression for the distribution function of the trapped particles is given for the case when the distribution function of the untrapped particles is a step function. The relation between δ and ψ for this case is established. The last section summarizes the results.

1. Mathematical Formulation

In order to investigate the properties of solitary BGK waves we start with the set of Vlasov–Poisson equations for the electrostatic case. These equations are

$$\left.\begin{aligned} &V\frac{\partial}{\partial x}f_j(x,v)+\frac{e_j}{m_j}E(x)\frac{\partial}{\partial v}f_j(x,v)=0,\\ &\frac{d^2\phi}{dx^2}=4\pi n_0\sum_{j=i,e}e_j\int_{-\infty}^{\infty}f_j(x,v),\\ &E=-\partial\phi/\partial x.\end{aligned}\right\} \tag{1.1}$$

Equations (1.1) have been written for the one-dimensional case in a frame of reference moving with the wave. Here f_j is the distribution function for the ions and the electrons only ($j = i, e$), ϕ is the electrostatic potential, E the electric field and n_0 is the unperturbed plasma density.

We are interested in processes for which the characteristic time-scales are comparable with the period of electron plasma oscillations. Hence the motion of the ions can be neglected and the Poisson equation becomes

$$\frac{d^2\phi}{dx^2}=4\pi en_0\left[\int_{-\infty}^{\infty}f(x,v)\,dv-1\right], \tag{1.2}$$

where $f(x,v)$ is the distribution function of the electrons.

Since the energy of an electron in an electrostatic field is an integral of the motion,

$$E=\tfrac{1}{2}mv^2-e\phi=\text{const.},$$

we can change from the variable v to E using

$$v=\operatorname{sgn}\left(\frac{e}{m}(E+e\phi)\right)^{\frac{1}{2}},\quad \frac{dv}{dE}=\frac{\operatorname{sgn}v}{(2m(E+e\phi))^{\frac{1}{2}}}.$$

Now equation (1·2) can be rewritten as

$$\frac{d^2\phi}{dx^2}=4\pi en_0\left\{\int_{-e\phi}^{\infty}dE\frac{[f^{(+)}(E)+f^{(-)}(E)]}{(2m(E+e\phi))^{\frac{1}{2}}}-1\right\}, \tag{1.3}$$

where $f^{(+)}(E)$ and $f^{(-)}(E)$ correspond to $v > 0$ and $v < 0$, respectively. This enables us to take into account any possible asymmetry in the distribution function.

We assume, however, that the potential has a symmetric form corresponding to experimental results and to the results of computer simulations (see Lynov *et al.* 1969; Turikov 1978). For such a case the electrons can be divided into two groups: (i) for $E < 0$, trapped electrons, which oscillate in the area of the localized wave; (ii) for $E \geqslant 0$, untrapped electrons.

Equation (1.3) can now be rewritten as

$$\frac{d^2\phi}{dx^2} = 4\pi e n_0\left\{\int_{-e\phi}^{0} dE\,\frac{[f_{\rm tr}^{(+)}(E)+f_{\rm tr}^{(-)}(E)]}{(2m(E+e\phi))^{\frac{1}{2}}}+\int_0^\infty dE\,\frac{[f_p^{(+)}(E)+f_p^{(-)}(E)}{(2m(E+e\phi))^{\frac{1}{2}}}-1\right\}, \qquad (1.4)$$

where $f_{\rm tr}^{(\pm)}(E)$ is the distribution function of the trapped electrons corresponding to $E < 0$ and $f_p^{(\pm)}(E)$ is the distribution function of the untrapped electrons corresponding to $E \geqslant 0$.

From the Vlasov equation (1.1) we see that $f_{\rm tr}^{(+)}(E) = f_{\rm tr}^{(-)}(E)$ which corresponds to a symmetric distribution function for the trapped electrons. Equation (1.4) can be written in the form of an Abel-type integral equation:

$$g(e\phi) = \frac{1}{2}\left\{\frac{1}{4\pi e n_0}\frac{d^2\phi}{dx^2}-\int_0^\infty dE\,\frac{[f_p^{(+)}(E)+f_p^{(-)}(E)]}{(2m(E+e\phi))^{\frac{1}{2}}}+1\right\} \qquad (1.5)$$

where

$$g(e\phi) = \int_{-e\phi}^{0}\frac{dE f_{\rm tr}(E)}{(2m(E+e\phi))^{\frac{1}{2}}}$$

and is a known function which can be determined from $f_p^{\pm}(E)$ and the dependence of $d^2\phi/dx^2$ on ϕ.

The Abel integral equation has the solution (Smirnov 1958)

$$f_{\rm tr}(E) = \frac{2m}{\pi^{\frac{1}{2}}}\int_0^{-E}\frac{dg}{dV}\frac{dV}{(-E-V)^{\frac{1}{2}}}, \qquad (1.6)$$

where $V = e\phi$ and

$$\frac{dg}{dV} = \frac{dQ}{dV}+\frac{1}{4(2m)^{\frac{1}{2}}}\int_0^\infty dE\,\frac{[f_p^{(+)}(E)+f_p^{(-)}(E)]}{(E+V)^{\frac{3}{2}}} \qquad (1.7)$$

where

$$Q = \frac{1}{8\pi e n_0}\frac{d^2\phi}{dx^2}.$$

The distribution function of the trapped electrons can be written as

$$f_{\rm tr}(E) = \frac{(2m)^{\frac{1}{2}}}{2\pi}\int_0^{-E}\frac{dQ}{dV}\frac{dV}{(-E-V)^{\frac{1}{2}}}+\frac{1}{4(2m)^{\frac{1}{2}}}\int_0^{-E}\int_0^\infty dV\,dE'\,\frac{[f_p^{(+)}(E')+f_p^{(-)}(E')]}{(-E-V)^{\frac{1}{2}}(E'+V)^{\frac{3}{2}}}. \qquad (1.8)$$

Integrating over V in the second term on the right-hand side of (1.7), we finally obtain

$$f_{\rm tr}(E) = \frac{(2m)^{\frac{1}{2}}}{\pi}\int_0^{-E}\frac{dQ}{dV}\frac{dV}{(-E-V)^{\frac{1}{2}}}+\frac{(-E)^{\frac{1}{2}}}{2\pi}\int_0^\infty dE'\,\frac{[f_p^{(+)}(E')+f_p^{(-)}(E')]}{(E')^{\frac{1}{2}}(E'-E)}. \qquad (1.9)$$

Equation (1.8) can now be used to find an expression for the distribution function of the trapped electrons ($E < 0$) if the electric potential ϕ and the distribution function of the untrapped particles are known.

2. Distribution function of the trapped electrons

The profile of the electric potential is taken as

$$\phi(x) = \phi_0\,\mathrm{sech}^2\,2x/l. \qquad (2.1)$$

Expression (2.1) is a soliton for a particular relationship between ϕ_0 the amplitude and l the width. We consider ϕ_0 and l to be free parameters and will establish a relationship between them assuming the condition that the distribution function of the trapped particles is non-negative. This form of the potential (equation (2.1)) fully corresponds with experimental results (Lynov *et al.* 1979) and computer simulation results (Turikov 1978). In the above mentioned works the solitary BGK wave was observed to have a symmetrical bell-shaped form. Using (2.1) and (1.7) we find that

$$\frac{dQ}{dV} = -\frac{1}{E_l}\left(1-\frac{3V}{V_0}\right),$$

where $E_l = \frac{1}{8}m\omega_p^2 l$, $V_0 = e\phi_0$ and ω_p is the frequency of the electron plasma oscillations. Integrating the first term on the right-hand side of (1.9), we get

$$\frac{(2m)^{\frac{1}{2}}}{\pi}\int_0^{-E}\frac{dQ}{dV}\frac{dV}{(-E-V)^{\frac{1}{2}}} = \frac{(2m)^{\frac{1}{2}}}{\pi E_l}2(-E)^{\frac{1}{2}}\left(1+\frac{2E}{V_0}\right). \tag{2.2}$$

We first compute $f_{\text{tr}}(E)$ when the distribution function of the untrapped electrons is given by a step distribution function which is normally used in the 'water bag' model (Berk & Roberts 1967):

$$\left.\begin{aligned} f_p^{(+)}(E) &= \frac{1}{2v_{\text{th}}}\theta(1-M-(E/T)^{\frac{1}{2}}),\\ f_p^{(-)}(E) &= \frac{1}{2v_{\text{th}}}\theta(1+M-(E/T)^{\frac{1}{2}})\,\theta(1-M+(E/T)^{\frac{1}{2}}),\end{aligned}\right\} \tag{2.3}$$

where $v_{\text{th}} = (2T/m)^{\frac{1}{2}}$, and T is the electron temperature and $M = v_0/v_{\text{th}}$. $\theta(x)$ is the Heavyside function.

$$\theta(x) = \begin{cases} 1, & x \geqslant 0,\\ 0, & x < 0.\end{cases}$$

The quantity M is in some sense an anologue of the Mach number for solitons (in which case $M = v_0/v_{\text{ph}}$ where v_{ph} is the phase velocity of the wave). However in our case M can be larger and less than unity.

Using (2.3) for integrating the second term on the right-hand side of (1.9), we get

$$\begin{aligned} &\frac{(-E)^{\frac{1}{2}}}{2\pi}\int_0^\infty dE'\frac{[f_p^{(+)}(E')+f_p^{(-)}(E')]}{(E')^{\frac{1}{2}}(E'-E)}\\ &= \frac{(-E)^{\frac{1}{2}}}{2\pi v_{\text{th}}}\left[\operatorname{sgn}(1-M)\int_0^{T(1-M)^2}\frac{dE'}{(E')^{\frac{1}{2}}(E'-E)}+\int_0^{T(1+M)^2}\frac{dE'}{(E')^{\frac{1}{2}}(E'-E)}\right]\\ &= \frac{1}{2\pi v_{\text{th}}}\left\{\tan^{-1}\left[\left(-\frac{T}{E}\right)^{\frac{1}{2}}(1-M)\right]+\tan^{-1}\left[\left(-\frac{T}{E}\right)^{\frac{1}{2}}(1+M)\right]\right\}. \end{aligned} \tag{2.4}$$

Using (1.9), (2.2) and (2.4), we find that

$$f_{\text{tr}}(W) = \frac{1}{\pi v_{\text{th}}}\left\{\frac{32W^{\frac{1}{2}}}{\delta^2}\left(1-2\frac{W}{\psi}\right)+\frac{1}{2}\left[\tan^{-1}\frac{(1-M)}{W^{\frac{1}{2}}}+\tan^{-1}\frac{(1+M)}{W^{\frac{1}{2}}}\right]\right\} \tag{2.5}$$

where $W = -E/T$, $\delta = l/\lambda_d$ (λ_d is the electron Debye radius) and $\psi = e\phi_0/T$. It may be noted that δ and ψ are the dimensionless width and amplitude of the BGK wave, respectively.

We now proceed to compute $f_{tr}(W)$ when the distribution of the untrapped electrons is given by a Maxwellian.

$$f_p^{(\pm)}(E) = \frac{1}{\pi v_{th}} \exp[-(E/T)^{\frac{1}{2}} \pm M)^2]. \tag{2.6}$$

After algebraic manipulations similar to those for the step distribution function, we get

$$f_{tr}(W) = \frac{1}{\pi v_{th}} \left\{ \frac{32\, W^{\frac{1}{2}}}{\delta^2}\left(1 - 2\frac{W}{\psi}\right) + \frac{1}{\pi}[I(W^{\frac{1}{2}}, M) + I(W^{\frac{1}{2}}, -M)] \right\}, \tag{2.7}$$

where

$$I(\alpha, \beta) = \int_0^\infty \frac{e^{-(\alpha x + \beta)^2}}{1+x^2}\, dx.$$

The integral $I(\alpha, \beta)$ cannot be determined analytically except for the case $\beta = 0$ ($M = 0$). For $M = 0$ we obtain

$$f_{tr}(W) = \frac{1}{\pi v_{th}} \left\{ \frac{32}{\delta^2} W^{\frac{1}{2}}\left(1 - 2\frac{W}{\psi}\right) + \pi^{\frac{1}{2}} e^W [1 - \Phi(W^{\frac{1}{2}})] \right\}, \tag{2.8}$$

where

$$\Phi = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x e^{-y^2}\, dy$$

is the probability integral. Expression (2.8) corresponds to the stationary BGK waves which have been observed in computer simulation experiments (Lynov *et al.* 1979; Turikov 1978).

3. Analysis of the dependence of the width of the BGK wave on its amplitude

Using expressions obtained in the preceding section we can now establish for what values of δ and ψ will $f_{tr}(W)$ be non-negative. Expressions (2.5), (2.7) and (2.6) consist of two terms each on the right-hand side. The second term is always positive. The first term

$$\frac{32}{\delta^2} W^{\frac{1}{2}}\left(1 - 2\frac{W}{\psi}\right) \tag{3.1}$$

which is determined from the form of the potential (2.1) changes its sign as W varies from 0 to ψ ($W = (-\frac{1}{2}mv^2 + e\phi)/T$) and $W_{max} = \psi$. Expression (3.1) has the largest negative value when $W = \psi$ which corresponds to the minimum of f_{tr}. Thus, the condition of non-negative $f_{tr}(W)$ can be expressed as

$$f_{tr}(\psi) \geqslant 0. \tag{3.2}$$

The case when

$$f_{tr}(\psi) = 0 \tag{3.3}$$

corresponds to observations in the computer simulation experiments (Lynov *et al.* 1979; Turikov 1978). So we shall investigate the relationship between δ and ψ for this case.

For a step distribution function for the untrapped particles, we find from (2.5) and (3.3) that

$$\delta = \frac{8\psi^{\frac{1}{4}}}{[\tan^{-1}(1-M)/\psi^{\frac{1}{2}} + \tan^{-1}(1+M)/\psi^{\frac{1}{2}}]^{\frac{1}{2}}}. \tag{3.4}$$

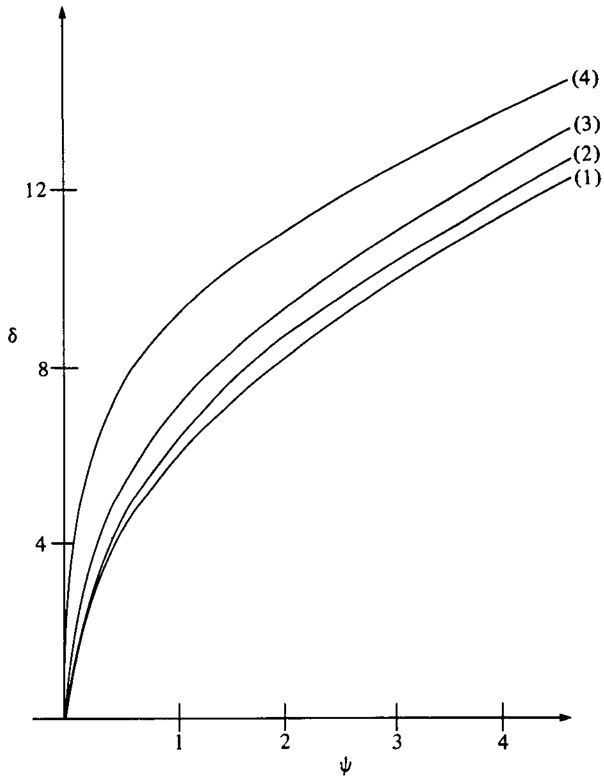

FIGURE 1. Dependence of width δ of a solitary BGK wave on its amplitude ψ. Curves (1)–(4) correspond to $M = 0$, $M = 0{\cdot}5$, $M = 1{\cdot}0$, and $M = 1{\cdot}5$, respectively. The untrapped particles have a step distribution function.

When $f_p^{(\pm)}(E)$ is Maxwellian we find, using (2.6), (2.7) and (3.3), that

$$\delta = \frac{2^{\frac{5}{2}}(\pi\psi)^{\frac{1}{4}}}{[I(\psi^{\frac{1}{2}}, M) + I(\psi^{\frac{1}{2}}, -M)]^{\frac{1}{2}}}, \tag{3.5}$$

and, for $M = 0$,

$$\delta = \frac{2^{\frac{5}{2}}(\pi\psi)^{\frac{1}{4}}}{[e^{\psi}(1 - \Phi(\psi^{\frac{1}{2}}))]^{\frac{1}{2}}}. \tag{3.6}$$

Expressions (3.4)–(3.6) correspond to the case of minimum width δ for a given value of the amplitude ψ.

We may also note that, for the case of a soliton solution, $\delta \propto \psi^{-\frac{1}{2}}$ which differs significantly from the relation between δ and ψ for the BGK wave. The reason for this difference is that for the soliton the electrons play a passive role, entering the equation via the Poisson equation. Thus one is free to choose the electron number density. However, in the case of the solitary BGK wave the electron density is not a free parameter and, additionally, electron hydrodynamic equations do not include trapping effects; for this reason the kinetic description via the Vlasov equation is necessary. (Lynov *et al.* 1979; Schamel 1979).

Figures 1 and 2 show the dependence of δ upon ψ for BGK waves. Figure 1 corresponds to the case when $f_p^{(\pm)}(E)$ is a step function; in figure 2, $f_p^{(\pm)}(E)$ is taken

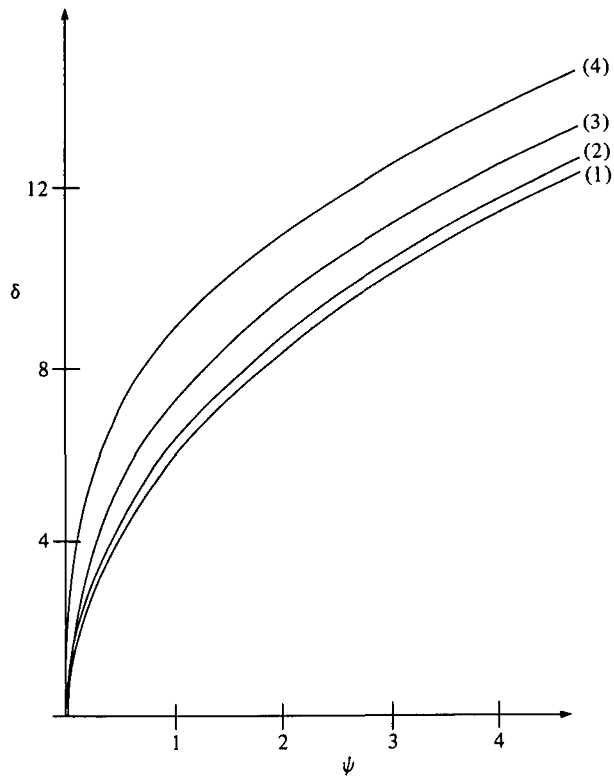

FIGURE 2. The same as figure 1 except that the untrapped particles have a Maxwellian distribution function.

to be a Maxwellian. It can be seen from figures 1 and 2 that dependence of δ upon ψ is not significantly affected by the distribution function of the untrapped particles, thus making it possible to use an analytically simple form for $f_p^{(\pm)}(E)$. We also note that with the increase of M (i.e. increase in velocity v_0 of the wave) the width δ increases. However, for amplitudes $\psi \gtrsim 2$ the curves tend to flatten out.

4. BGK waves in a cylindrical waveguide containing a magnestized plasma

Computer simulations (Lynov *et al.* 1979) have been used to study the excitation of solitary BGK waves in cylidrical waveguides. The magnetic field is along the x axis of the cylindrical waveguide and waves are considered to propagate along this axis only.

Poissons equation in cylindrical co-ordinates has the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial\gamma}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 4\pi e(n-n_0)$$

where $n = n(x,r,\theta)$ is the electron density at a given point and n_0 is the unperturbed plasma density.

We assume that the potential at the surface of the waveguide is zero:

$$\phi(x, r_0) = 0$$

where r_0 is the radius of the waveguide. For such boundary conditions ϕ and n are expressed as

$$\phi(x,r,\theta) = \sum_{\mu\nu} \phi_{\mu\nu}(x)\, J_\nu\left(p_{\mu\nu}\frac{r}{r_0}\right)e^{i\nu\theta},$$

$$n(x,r,\theta) = \sum_{\mu\nu} n_{\mu\nu}(x)\, J_\nu\left(p_{\mu\nu}\frac{r}{r_0}\right)e^{i\nu\theta},$$

where J_ν is the first- order Bessel function and $p_{\mu\nu}$ is the μth root of J_ν. We further assume that only the zeroth radial mode $J_0(p_{00}\, r/r_0)$ is excited (see, for example, Trivelpiece & Gould 1959; Manheimer 1969). Thus Poisson's equation can be rewritten as

$$\frac{d^2\phi_{00}(x)}{dx^2} = k_\perp^2\, \phi_{00}(x) + 4\pi e n_{00}(x) \tag{4.1}$$

where

$$\kappa_\perp = \frac{p_{00}}{r_0} \simeq \frac{2{\cdot}404}{r_0}.$$

Under the assumptions made, the distribution function of the electrons can be written as

$$f(x,r,v) = F_0(v) + f_{00}(x,v)\, J_0(k_\perp r) \tag{4.2}$$

where $F_0(v)$ is the unperturbed distribution function. The potential ϕ can be expressed as

$$(\phi x, r) = \phi_{00}(x)\, J_0(k_\perp r). \tag{4.3}$$

Equations (4.2) and (4.3) are substituted in the Vlasov equation for the stationary case and we obtain

$$v\frac{\partial f_{00}}{\partial x} J_0(k_\perp r) + \frac{e}{m}\frac{d\phi_{00}}{dx} J_0(k_\perp r)\frac{\partial}{\partial v}[F_0(v) + f_{00} J_0(k_\perp r)] = 0. \tag{4.4}$$

Multiplying this equation by $rJ_0(r)$ and integrating over r from 0 to r_0, we get (Lynov *et al.* 1979)

$$v\frac{\partial f_{00}}{\partial x} + \frac{e}{m}\frac{d\phi_{00}}{dx}\frac{\partial F_0}{\partial v} + \alpha\frac{e}{m}\frac{d\phi_{00}}{dx}\frac{\partial f_{00}}{\partial v} = 0 \tag{4.5}$$

where

$$\alpha = \frac{\int_0^{r_0} rJ_0^3(k_\perp r)\, dr}{\int_0^{r_0} rJ_0^2(k_\perp r)\, dr} \simeq 0.72.$$

The above expression can be written in more standard form for a one-dimensional Vlasov equation, if we use the substitution

$$f(x,v) = F_0(v) + \alpha f_{00}, \quad \tilde{\phi} = \alpha\phi_{00}(x).$$

As a result, (4.5) and (4.1) respectively can be written as

$$v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{d\tilde{\phi}}{dx}\frac{\partial f}{\partial v} = 0, \tag{4.6}$$

$$\frac{d^2\tilde{\phi}}{dx^2} - k_\perp^2\,\tilde{\phi} = 4\pi e n_0\left[\int_{-\infty}^{\infty} f(x,v)\, dv - 1\right], \tag{4.7}$$

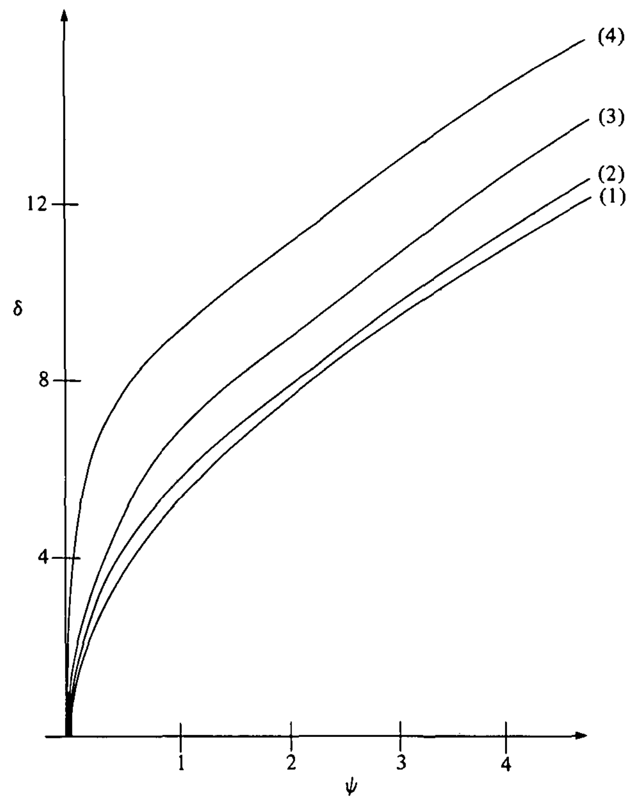

FIGURE 3. The same as figure 1 except that the untrapped particles have a step distribution function and the plasma is in a cylindrical waveguide.

It may be noted that the distribution function f has been normalized to 1. We see from the above equations that a plasma in a cylindrical waveguide differs from an unbounded plasma by the presence of the term $-k_\perp^2\tilde{\phi}$ in (4.7).

The function $Q(V)$ (see (1.7)) now becomes

$$Q(V) = \frac{1}{8\pi e n_0}\left(\frac{d^2\tilde{\phi}}{dx^2} - k_\perp^2\tilde{\phi}\right).$$

In order to write down an expression for $f_{tr}(W)$, we repeat the same procedure as in § 1 and obtain

$$f_{tr}(W) = \frac{1}{\pi v_{th}}\left\{\frac{32W^{\frac{1}{2}}}{\delta^2}\left(1-2\frac{W}{\alpha\psi}\right)+\frac{1}{2}\left[\tan^{-1}\frac{(1-M)}{W^{\frac{1}{2}}}+\tan^{-1}\frac{(1+M)}{W^{\frac{1}{2}}}-\kappa_\perp^2 2W^{\frac{1}{2}}\right]\right\} \tag{4.8}$$

where $\kappa_\perp = k_\perp\lambda_d$.

The dependence of δ upon ψ for the case $f_{tr}(W) = 0$ is given by

$$\delta = \frac{8(\alpha\psi)^{\frac{1}{4}}}{\tan^{-1}((1-M)/(\alpha\psi)^{\frac{1}{2}})+\tan^{-1}((1+M)/(\alpha\psi)^{\frac{1}{2}})-\kappa_\perp^2 4(\alpha\psi)^{\frac{1}{2}}]^{\frac{1}{2}}}. \tag{4.9}$$

From (4.9) we note that the width of the solitary BGK wave increases with a decrease in r_0 (increase in $\kappa_\perp$) for a given value of ψ. Figure 3 shows the dependence of δ upon ψ for parameters for a step distribution function for $f_p^{(\pm)}$. The numerical values of δ and ψ correspond very well with those in the computer simulation experiments (see Lynov *et al.* 1979; Turikov 1978).

From equation (4.9) we see that, owing to the presence of the factor $-4\kappa_{\perp}^2(\alpha\psi)^{\frac{1}{2}}$ in the denominator, there is an upper limit on the maximun value of ψ. For the case $M = 0$, when $(\alpha\psi_{\text{max}})^{\frac{1}{2}} \simeq 1$ the maximum value for ψ is given by

$$\psi_{\text{max}} \simeq \tfrac{1}{2}\alpha\kappa_{\perp}^2.$$

The physical explanation of this upper limit is that for small values of r_0 longitudinal perturbations in the density of the plasma will contribute mainly to perturbations in the radial component of the electric field. Thus a small longitudinal component cannot support a stationary distribution of electrons in the region of the BGK wave.

5. Conclusions

Following the method of Bernstein, Greene and Kruskal (1957), we have investigated some properties of stationary BGK waves. The main results of the work can be summarized as follows.

First, we obtained expressions for the distribution functions of the trapped particles for the cases of step and Maxwellian distribution of the untrapped particles.

Secondly, using the condition of a non-negative distribution function of the trapped particles we found the relationship between the amplitude and the width of the solitary BGK wave. The difference between this relation and that for a soliton solution was noted.

Thirdly, we also investigated the properties of the BGK waves propagating along the magnetic field for a plasma in a cylindrical wave-guide. The relation between the maximum possible amplitude of the BGK wave and the radius of the cylindrical waveguide was also established.

Fourthly it was noted that these results are in close agreement with the results of computer simulation experiments.

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