J. Plasma Physics (1984), *vol.* 31, *part* 2, *pp.* 225–229
Printed in Great Britain

Excitation of whistler mode instability due to slow cyclotron interaction in an inhomogenous beam–plasma system

By H. A. SHAH† and V. K. JAIN

Plasma and Space Physics Group, School of Mathematical and Physical Sciences, University of Sussex, Brighton, BN1 9QH

(Received 22 July 1983 and in revised form 17 October 1983)

The excitation of whistler wave instability due to slow cyclotron ($m = -1$) interaction in an inhomogeneous plasma penetrated by an inhomogeneous beam of electrons is studied. Expressions are obtained for the elements of the plasma and beam dielectric tensors. It is shown that the inhomogeneity in both beam and plasma number densities affects the growth rate of the instability.

In a recent paper, Shah & Jain (1983) (henceforth referred to as I) investigated the whistler mode instability in an inhomogeneous plasma in the presence of an inhomogeneous beam of electrons. The effect of the inhomogeneity in the plasma and beam number densities on the growth rate and the frequency of the instability excited due to Čerenkov interaction (characterized by the condition $\omega - k_z v_b = m\omega_{ce}$; $m = 0$) was studied. In this note, as an extension to the above study, we consider the excitation of the instability due to slow cyclotron ($m = -1$) interaction. This involves the coupling of the whistler mode wave to the slow cyclotron wave on the beam. We study the effect of the inhomogeneity on the frequency and growth rate of the instability.

As in I, both beam and plasma are considered weakly inhomogeneous, with the density gradient in the y direction. The equilibrium distribution functions for the beam and plasma are taken to be anisotropic Maxwellians. The external magnetic field is in the z direction. The contribution of the ions to the dielectric tensor is neglected since for $m = -1$ interactions the instability frequencies are much higher than the ion cyclotron or ion plasma frequencies.

Following the method outlined in I and noting that, for $m = -1$, $\mathbf{q}$ and $\mathbf{p}$ (see equations (4) and (6) of I), are given by

$$\mathbf{q} = \left(v_{\perp 0\alpha}\frac{J_1}{\xi}, v_{\perp 0\alpha}J_1', v_{z0\alpha}J_1\right),$$

$$\mathbf{p} = \left(v_{\perp 0\alpha}^2(J_1 + J_1''), -iv_{\perp 0\alpha}^2\left(\frac{J_1}{\xi}\right)', v_{\perp 0\alpha}v_{z0\alpha}\frac{J_1}{\xi}\right)$$

† Present address: Department of Applied Mathematics, Queen Mary College, London E1 4NS.

where $J_1 = J_1(\xi)$ is the first-order Bessel function, $\xi = k_z v_{\perp 0\alpha}/\omega_{ce}$ and α refers to the beam or the plasma, we obtain expressions for the elements of the dielectric tensor.

The dielectric tensor elements of the plasma are

$$\epsilon^p_{xx} = 1 - \frac{2a(A+B)\,g'_1}{\alpha(\alpha+1)\,\xi_p^2}, \qquad \epsilon^p_{xy} = -\frac{2ia(A+B)\,g'_2}{\alpha(\alpha+1)\,\xi_p^2}, \qquad \epsilon^p_{xz} = -\frac{4a(A+B)\,g'_3}{\eta\xi_p^2\alpha},$$

$$\epsilon^p_{yx} = \frac{2ia(A+B)\,g'_4}{\alpha(\alpha+1)\,\xi_p^2}, \qquad \epsilon^p_{yy} = 1 - \frac{2a(A+B)\,g'_5}{\alpha(\alpha+1)\,\xi_p^2}, \qquad \epsilon^p_{yz} = \frac{4ia(A+B)\,g'_6}{\eta\xi_p^2\alpha},$$

$$\epsilon^p_{zx} = \frac{4a(A+B)\,g'_7}{\eta\xi_p^2\alpha}, \qquad \epsilon^p_{zy} = -\frac{4ia(A+B)\,g'_8}{\eta\xi_p^2\alpha}, \qquad \epsilon^p_{zz} = 1.$$

Here

$$A = 2(1+1/g), \quad B = \xi_p\psi_p/\alpha,$$

$$g'_1 = \Lambda_p + \tfrac{1}{8}\xi_p^3\psi_p[{}_1F_1(\tfrac{3}{2};2;-\xi_p^2) + \tfrac{1}{8}\xi_p^2\,{}_2F_2(\tfrac{5}{2},3;2,5;-\xi_p^2)],$$

$$g'_2 = \tfrac{1}{4}\xi_p^2(1-\psi_p/\xi_p)\,[\tfrac{1}{4}\xi_p^2\,{}_1F_1(\tfrac{5}{2};4;-\xi_p^2)\,{}_1F_1(\tfrac{3}{2};2;-\xi_p^2)] - (\psi_p/\xi_p)\,\Lambda_p,$$

$$g'_3 = g'_1, \quad g'_4 = g'_2 + 2(\psi_p/\xi_p)\,\Lambda_p,$$

$$g'_5 = \tfrac{1}{4}(1+\psi_p/\xi_p)\,[\tfrac{3}{32}\xi_p^4\,{}_2F_2(\tfrac{5}{2},4;3,5;-\xi_p^2) - \tfrac{1}{2}\xi_p^2\,{}_2F_2(\tfrac{3}{2},2;1,3;-\xi_p^2)$$
$$+ \tfrac{1}{2}(\psi_p/\xi_p)\,\{\tfrac{1}{4}\xi_p^2\,{}_1F_1(\tfrac{5}{2};4;-\xi_p^2) - {}_1F_1(\tfrac{3}{2};2;-\xi_p^2)\}],$$

$$g'_6 = \xi_p^2(1+\psi_p/\xi_p)\,[\tfrac{1}{4}\xi_p^2\,{}_1F_1(\tfrac{5}{2};4;-\xi_p^2) - {}_1F_1(\tfrac{3}{2};2;-\xi_p^2)] + (\psi_p/\xi_p)\,\Lambda_p$$

$$g'_7 = \Lambda_p(1+\psi_p/\xi_p), \quad g'_8 = g'_6 - (\psi_p/\xi_p)\,\Lambda_p,$$

$$\xi_p = \frac{k_x v_{T\perp p}}{\omega_{ce}}, \quad \frac{T_{\parallel ep}}{T_{\perp ep}} = g, \quad \psi_p = \frac{v_{T\perp p}}{L_p\,\omega_{ce}}, \quad \frac{\omega_{pe}^2}{\omega_{ce}^2} = a,$$

$$L_p = \left(\frac{1}{n_p}\frac{\partial n_p}{\partial y}\right)^{-1}, \quad \alpha = \frac{\omega}{\omega_{ce}}, \quad \eta = \frac{k_z}{k_x}$$

and

$$\Lambda_p = \exp(-\tfrac{1}{2}\xi_p^2)\,I_1(\tfrac{1}{2}\xi_p^2).$$

I_1 is the first-order Bessel function of an imaginary argument.

The elements of the dielectric tensor of the beam are given by

$$\epsilon^b_{xx} = 1 + \frac{2a\theta g_1}{\alpha\eta d_1^{\frac{1}{2}} d_2\,\xi_p^3}(D_r + iD_i), \quad \epsilon^b_{xy} = -\frac{2a\theta\,g_2}{\alpha\eta d_1^{\frac{1}{2}} d_2\,\xi_p^3}(iD_r - D_i),$$

$$\epsilon^b_{xz} = -\frac{2a\theta}{\alpha\eta d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}\,\xi_p^2}\{\pi^{\frac{1}{2}} d_2^{\frac{1}{2}} b^{\frac{1}{2}}\xi_p\psi_p + g_3(E_r + iE_i)\}, \quad \epsilon^b_{yx} = \frac{2a\theta\,g_4}{\alpha\eta d_2^{\frac{1}{2}} d_2\,\xi_p^3}(D_r + iD_i),$$

$$\epsilon^b_{yy} = 1 + \frac{2a\theta\,g_5}{\alpha\eta d_1^{\frac{1}{2}} d_2\,\xi_p^3}(D_r + iD_i), \quad \epsilon^b_{yz} = \frac{2a\theta\,g_6}{\alpha\eta d_2\xi_p^2}(E_r - iE_i),$$

$$\epsilon^b_{zx} = -\frac{2a\theta}{\alpha\eta d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}\,\xi_p^2}\left\{\frac{\xi_p^2\eta b^{\frac{1}{2}} d_2\psi_p}{\alpha} + g_7(E_r + iE_i)\right\}, \quad \epsilon^b_{zy} = -\frac{2a\theta\,g_6}{\alpha\eta\,d_2^{\frac{1}{2}}\,\xi_p}(iE_r - E_i),$$

$$\epsilon^b_{zz} = 1 + \frac{2a\theta\,d_1^{\frac{1}{2}}}{\alpha\eta d_2\,\xi_p}\left\{\frac{2\eta d_1^{\frac{1}{2}}\,\xi_p}{\alpha}\left(\frac{1}{2} + \frac{2b}{d_1}\right) + g_9(G_r + iG_i)\right\}.$$

Here

$$D_r = -2\eta\xi_p/\alpha d_1^{\frac{1}{2}}, \quad D_i = (A_b+B_b)\pi^{\frac{1}{2}},$$

$$E_r = (A_b+B_b) - C(\alpha+1)/\eta d_1^{\frac{1}{2}}\xi_p, \quad E_i = \frac{\alpha+1}{\eta d_1^{\frac{1}{2}}\xi_p} D_i,$$

$$G_r = (A_b+B_b)\frac{b^{\frac{1}{2}}}{d_1^{\frac{1}{2}}}\left\{1+\frac{(\alpha+1)}{\eta d_1^{\frac{1}{2}}\xi_p}\frac{d_1^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right\} - C\left(\frac{1}{2}+\left(\frac{(\alpha+1)}{\eta d_1^{\frac{1}{2}}\xi_p}\right)^2\right), \quad G_i = \left(\frac{\alpha+1}{\eta d_1^{\frac{1}{2}}\xi_p}\right)^2 D_i,$$

$$C = 2\zeta\xi_p/\alpha d_1^{\frac{1}{2}}, \quad A_b = 2(1+d_2/d_1)(1-\eta\xi_p b^{\frac{1}{2}}/\alpha),$$

$$B_b = d_2\xi_p\psi_p/\alpha, \quad g_1 = \Lambda_b + \tfrac{1}{8}\xi_p^3 d_2^2 \psi_b\, {}_1F_1(\tfrac{3}{2}; 2; -\xi_p^2 d_2),$$

$$g_2 = \tfrac{1}{4}\xi_p^2 d_2(1-\psi_b/\xi_p)\left[\tfrac{1}{4}\xi_p^2 d_2\, {}_1F_1(\tfrac{5}{2}; 4; -\xi_p^2 d_2) - {}_1F_1(\tfrac{3}{2}; 2; -\xi_p^2 d_2)\right], \quad g_3 = g_1,$$

$$g_4 = \tfrac{1}{4}\xi_p^2 d_2(1+\psi_b/\xi_p)\left[\tfrac{1}{4}\xi_p^2 d_2\, {}_1F_1(\tfrac{5}{2}; 4; -\xi_p^2 d_2) - {}_1F_1(\tfrac{3}{2}; 2; -\xi_p^2 d_2)\right], \quad g_6 = g_4,$$

$$g_7 = \Lambda_b(1+\psi_b/\xi_p), \quad g_8 = g_4 - (\psi_b/\xi_p)\Lambda_b, \quad g_9 = g_7,$$

$$\theta = n_{0b}/n_{0p}, \quad d_1 = T_{\parallel b}/T_{\perp p}, \quad d_2 = T_{\perp b}/T_{\perp p},$$

$$b = v_b^2/v_{T\perp p}^2, \quad \psi_b = v_{T\perp p}/L_b\omega_{ce}, \quad L_b = \left(\frac{1}{n_{0b}}\frac{\partial n_{0b}}{\partial y}\right)^{-1}$$

and

$$\Lambda_b = \exp(-\tfrac{1}{2}\xi_p^2 d_2)\, I_1(\tfrac{1}{2}\xi_p^2 d_2).$$

In the above set of expressions the subscripts or superscripts p, b refer to the plasma or beam respectively. $L_{p,b}$ is the scale length of the inhomogeneity. $v_{T\perp p}$, $v_{T\parallel p}$, $v_{T\perp b}$, $v_{T\parallel b}$ are the perpendicular and parallel plasma and beam electron thermal velocities. v_b is the velocity of the beam along the z direction. ${}_jF_k$ are the hypergeometric series and k_x and k_z are the perpendicular and parallel wavenumbers. ω, ω_p, ω_b, ω_{ce} are the wave, plasma, beam and electron cyclotron frequencies respectively.

It should be noted that while writing ϵ_{ij}^b we have replaced the Fried–Conte dispersion function by a power series (using equation (20) of I) as the argument of the function is very small around resonance ($\omega - k_z v_b + \omega_{ce} = 0$). However, in the case of the plasma dielectric tensor, we use an asymptotic expansion (equation (21) of I) for the Fried–Conte dispersion function. This is because around resonance the argument of the Fried–Conte dispersion function is large since the beam drift velocity is larger than the parallel electron plasma thermal velocity.

Having obtained the dielectric tensor elements for the beam and the plasma, the general dispersion relation can now be written as

$$D = D_p + D_b \tag{1}$$

where D_p and D_b are the dispersion relations of the beam and plasma respectively and are given by (13), (15), (16) and (17) of I. The growth rate can then be calculated using the expression

$$\gamma = -\operatorname{Im} D_b(\mathbf{k}, \alpha_r) \bigg/ \left[\frac{\partial}{\partial \alpha_r} D_p(\mathbf{k}, \alpha_r)\right]. \tag{2}$$

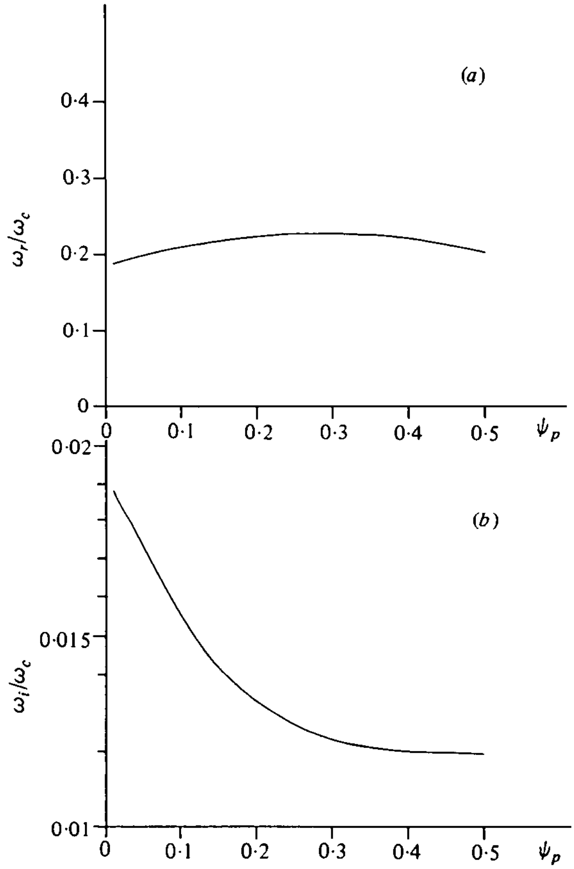

FIGURE 1. The variation of (a) the frequency (ω_r/ω_{ce}) and (b) the growth rate (ω_i/ω_{ce}) with the plasma inhomogeneity parameter ψ_p. Other plasma and beam parameters are $\omega_p/\omega_{ce} = 0{\cdot}3, k_x v_{T\perp p}/\omega_{ce} = 2$, $v_{T\perp p}/v_{T\perp p} = 1{\cdot}0$, $\omega_b/\omega_{ce} = 0{\cdot}05$, $v_{T\perp b}/v_{T\perp p} = v_{T\parallel b}/v_{T\perp p} = 1{\cdot}0$, $v_{T\perp p}/L_b\omega_{ce} = 0{\cdot}5$, $v_b/v_{T\perp p} = 4$, $1/\eta = 8{\cdot}1$.

We write down an expression for the growth rate for the electrostatic case as these instabilities are most easily excited.

$$\gamma_{es} = \frac{-\alpha\xi_p\theta\left[\dfrac{1}{d_1^{\frac{1}{2}}\xi_p}\left\{\dfrac{g_1 D_i}{\eta\xi_p} - E_i(g_3-g_7) - \dfrac{\psi_b}{\eta\xi_p}\left(g_4 D_r - \dfrac{g_6 E_r}{(1+\eta^2)^{\frac{1}{2}}}\right)\right\}\right]}{(A+B)\,d_2^{\frac{1}{2}}\left[\dfrac{2\alpha+1}{(\alpha+1)^2}g_1' + 2(g_3'-g_7') - \dfrac{\psi_p}{\xi_p}\left\{\dfrac{(2\alpha+1)}{(\alpha+1)^2}g_4' - \dfrac{2g_6'}{\eta(1+\eta^2)^{\frac{1}{2}}}\right\}\right]}. \tag{3}$$

Equation (3) is further simplified since we use a local approximation which requires that the projection of the wavelengths of the excited waves in the direction perpendicular both to the external magnetic field and to the direction of the density gradient must be greater than the electron Larmor radius, i.e. $k_x v_{T\perp p,b}/\omega_{ce} > 1$. Thus from Luke (1975, ch. 7) we have

$$F(a;c;u) \sim \frac{\Gamma(c)\,e^4 u^{a-c}}{\Gamma(a)}; \quad u \gg 1$$

and

$${}_2F_2 \ll {}_1F_1.$$

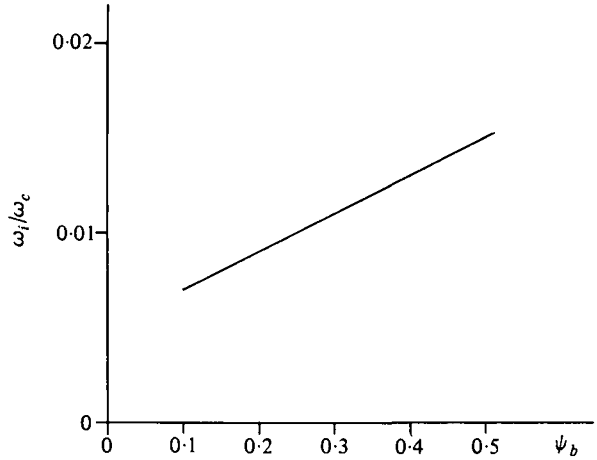

FIGURE 2. The variation of the growth rate (ω_i/ω_{ce}) of the whistler instability with ψ_b as parameter. Other parameters are as in figure 1 except for $v_{T\perp p}/L_r\omega_{ce} = 0{\cdot}1$.

We now study the effect of the inhomogeneity on the excitation of the whistler instability. Figures 1 (*a*) and (*b*) show the variation in the frequency and the growth rate of the instability, for the electrostatic case, as ψ_p, the normalized plasma inhomogeneity parameters, are varied. It is found that the effect of increasing ψ_p is mainly to reduce the growth rate of the instability. The frequency α_r shows a small variation with ψ_p. The effect of varying the beam inhomogeneity parameter ψ_b on the growth rate is shown in figure 2. The variation in the frequency of the instability with ψ_b is found to be negligible, as expected, since $n_{0b} \ll n_{0p}$. It can be seen from figure 2 that the growth rate increases with ψ_b.

Finally we note that we cannot consider angles of propagation arbitrarily close to 90°. This is because at near perpendicular propagation the contribution of ions to the whistler mode becomes important. Thus the work presented above is restricted to angles of propagation such that

$$\frac{\eta}{(1+\eta^2)^{\frac{1}{2}}} > \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$$

which implies that the angles of propagation must be less than about 88°.

To summarize, the inhomogeneity in the plasma and beam number densities are found to affect mainly the growth rates of the instabilities due to slow cyclotron interaction. It should also be noted that the above model allows us to consider how the frequency and growth rate would vary with varying parallel and perpendicular temperatures of the plasma and the beam, since anisotropic Maxwellian distribution functions have been used for the plasma and the beam.

We would like to thank the referee for his comments.

REFERENCES

LUKE, Y. L. 1975 *Mathematical functions and their approximations*. Academic.
SHAH, H. A. & JAIN, V. K. 1983 *J. Plasma Phys.* **29**, 439.