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ORIGINAL PAPER

Effect of Trapping on Vortices in Plasma

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Abstract Microscopic trapping of electrons is considered in one- and two-dimensional potential wells (shallow and deep) and its effect on vortex formation is investigated by deriving modified Hasegawa Mima (HM) equations. Inhomogenieties in the number density and magnetic field are taken into account. The modified HM equations are analysed by considering bounce frequencies of the trapped particles. Solitary vortices are obtained via Kortweg deVries (KdV) type of equations and both exact and Sagdeev potential solutions are obtained. In general it is observed that trapping produces stronger non-linearities and this leads to the modification of the original HM equation.

Keywords Hasegawa-Mima equation · Trapped particles · Vortices · Solitary wave

Introduction

Beginning with the seminal work of Hasegawa and Mima [1], who generalized the work of Charney [2], on the formation of vortices in inviscid fluids, two-dimensional dynamics and the subsequent formation of vortices in plasmas has captured the attention of many authors. This was followed by a flurry of activity in the area of vortices in plasmas. Early work on vortices was summed up comprehensively in the review paper by Horton [3], whereas Nycander [4] reviewed work in both plasmas and geophysical flows. And more recently Tsintsadze et al. [5, 6] described a new concept of generation of vortex rings by strong electromagnetic radiations and by the laser wake fields.

One of the simplest equations that admit localized vortex structures is the Hasegawa–Mima (HM) equation describing a two-dimensional flow in a non-uniform low frequency plasma. Due to its simple form the HM-equation has been studied extensively both analytically [1] and numerically [7]. In addition to vortex solutions the HM-equation contains drift waves which are a result of the non-uniform number density. In the weak turbulence approach [8] three of the linear drift modes interact resonantly via paremetric interactions and are subsequently used in the HM equation. In strong turbulence, however, a more accurate description would be a superposition of linear modes and vortex structures [9]. Since both linear waves and non-linear vortex structures exist in the HM-equation, a study of plasma turbulence (e.g. vortex–wave interaction) in the frame of this equation is natural.

Starting with the work of Bernstien, Greene and Kruskal (BGK) [10] in 1957 it became known that trapped particles exert a significant effect on the non-linear dynamics of plasmas. In this pioneering work [10] trapping was considered directly by the wave itself. However trapping as a microscopic process was considered by Gurevich [11] in 1967 where the solution of the Vlasov equation along with Maxwell's equations was used. BGK modes were investigated by Schamel [12] in unmagnetized plasmas for electrostatic perturbations however Schamel and coworkers continued to refine and further study BGK modes and the ensuing distribution functions of the trapped particles over the last forty years, these investigations have been summed up in reviews [13, 14]. Based on the results of Schamel

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mentioned above coherent electric field structures in the magnetosphere were investigated in Ref. [15] using the drift kinetic equation and solitary wave structures and electrostatic broadband noise were considered in Ref. [16, 17].

On the other hand computer simulations [18] and experimental work [19] confirmed the existence of trapping of particles as a microscopic phenomena. In. [20, 21] the effect of trapping of particles on the propagation characteristics of ion acoustic solitons by using Maxwellian and non-Maxwellian distribution functions, respectively was investigated. It was seen in both cases that dynamics of the ion acoustic solitons are considerably modified when trapping is taken into account.

In the present work we consider some new aspects of vortex formation in a plasma consisting of ions and electrons with the effect of adiabatic trapping of electrons in one and two dimensions taken into account. We consider a drift ion acoustic wave which is the most basic electrostatic low-frequency drift mode in an inhomogeneous plasma with a magnetic field with a gradient in the direction perpendicular to the ambient magnetic field. The electrons are considered hot enough to neglect their magnetization [22] and subsequently their drift velocities are also neglected in comparison to the electron thermal velocity. The ions are, however, considered to be cold and magnetized and their drifts are taken to be significant.

The paper is organized as follows. In section "One-dimensional Potential Well" we give the basic mathematical formulation for one-dimensional trapping and derive a modified HM equation. Various solutions to the modified HM equation for both the shallow and deep well trapping are considered. In section "Two-dimensional Potential Well" we consider two-dimensional trapping and again a modified HM equation is derived and investigated. Finally in section "Disussion and Conclusion" we present a general conclusion and discussion.

One-dimensional Potential Well

We begin by considering the trapped particle distribution function in one dimension only. Following Lifshitz and Pitaevskii [23], we can obtain the distribution of electrons in the shallow well case when $e\Phi/T < 1$

$$n_e = n_0\left(1 + \frac{e\Phi}{T_e} - \frac{4}{3\sqrt{\pi}}\left(\frac{e\Phi}{T_e}\right)^{\frac{3}{2}}\right) \quad (1)$$

and for the deep well potential, i.e., for $e\Phi/T > 1$ we obtain,

$$n_e = 2n_0\sqrt{\frac{e\Phi}{\pi T_e}} \quad (2)$$

We note here that e, Φ, and T_e are the electronic charge, potential and electron temperature, respectively and the free electrons are considered to obey a Maxwell Boltzmann distribution function.

The equations now needed for a complete description of low frequency electrostatic drift waves in plasmas are the equations of motion and continuity for ions and expressions (1) and (2) for the electrons trapped in a shallow and deep potential well cases, respectively. The ion equations in the MKS units are

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_i\cdot\nabla\right)\mathbf{v}_i = \frac{e}{m_i}(\mathbf{E} + \mathbf{v}_i\times\mathbf{B}) - \frac{1}{m_i n_i}\nabla p_i + \mathbf{g} \quad (3)$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i v_i) = 0 \quad (4)$$

Here n_i, $\mathbf{v}_i$, m_i, p_i, $\mathbf{g}$ are the ion density, velocity, charge, mass, ion pressure and gravitational acceleration, respectively. We assume the plasma to be quasineutral and only ions are magnetized by an ambient magnetic field $\mathbf{B} = B_0(x)\hat{z}$ which is assumed to have a weak dependence in the x direction. We assume that $T_e \gg T_i$ so that we can neglect the ion pressure in Eq. 3 for the sake of simplicity.

Following the method elaborated in Weiland [24], which proceeds by taking the curl of Eq. 3 and using Eq. 4 we obtain in the absence of baroclinic pressure the following equation.

$$\frac{d}{dt}\left[\ln\left(\frac{\Omega_i + \Omega_{ci}}{n_i}\right)\right] = 0 \quad (5)$$

where $\mathbf{\Omega}_i$ is the vorticity which is defined as

$$\mathbf{\Omega}_i = \nabla\times\mathbf{v}_i$$

and $\Omega_{ci} = eB/m_i$ is the ion gyrofrequency. We further note that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_i\cdot\nabla$$

where v_i in the drift approximation [24] is taken as

$$\mathbf{v}_i = \mathbf{v}_e + \mathbf{v}_g$$

where $\mathbf{v}_e$ and $\mathbf{v}_g$ are the $\mathbf{E}\times\mathbf{B}$ and gravitational drifts given by the following expressions, respectively

$$\mathbf{v}_e = -\left(\frac{\nabla\Phi\times\hat{z}}{B_0}\right)$$

$$\mathbf{v}_g = \frac{m\mathbf{g}\times\mathbf{B}_0}{B_0^2}$$

Considering two-dimensional propagation, we take $\nabla = \hat{x}\frac{\partial}{\partial z} + \hat{y}\frac{\partial}{\partial y}$, and vorticity $\mathbf{\Omega}_i$ only in the z direction can be expressed as

$$\mathbf{\Omega}_i = (\nabla\times\mathbf{v}_i)\cdot\hat{z} \quad (6)$$

We use the quasineutrality condition for perturbed number densities (plasma approximation) i.e., $\delta n_i = \delta n_e = \delta n$, where $\delta n \ll n_0$ and n_0 is the unperturbed ion number density. Further by taking $\Omega_i \ll \Omega_{ci}$ [24] we can write Eq. 5 as

$$\frac{d}{dt}\left[\ln\frac{\Omega_{ci}}{n_0}+\frac{\Omega_i}{\Omega_{ci}}-\frac{\delta n}{n_0}\right]=0 \tag{7}$$

In the presence of the inhomogeneous magnetic field $B_0(x)$ in the z direction, Eq. 6 can be rewritten as

$$\Omega_i=\frac{\nabla^2\Phi}{B_0}-mg\frac{\partial}{\partial x}\left(\frac{1}{B_0}\right)+\frac{\partial}{\partial x}\left(\frac{1}{B_0}\right)\frac{\partial\Phi}{\partial x} \tag{8}$$

We now proceed to develop Eq. 7 for the shallow well case by comparing $n_e = n_o\left(1+\frac{\delta n}{n_o}\right)$ and Eq. 1 we obtain

$$\frac{\delta n}{n_o}=\frac{e\Phi}{T_e}-\frac{4}{3\sqrt{\pi}}\left|\frac{e\Phi}{T_e}\right|^{\frac{3}{2}} \tag{9}$$

using Eqs. 8 and 9 in 7, we obtain finally by ignoring higher derivatives of magnetic field as its inhomogeneity is considered to be weak.

$$\begin{aligned}&\left(\partial_t+\mathbf{v}_g\partial_y\right)\left(\varrho^2\nabla^2\Psi-\Psi\right)-\left(v_e^*+v_{\nabla B}\right)\partial_y\Psi\\&=\frac{\varrho^2}{B_0}(\nabla\Psi\times z)\cdot\nabla\left(\nabla^2\Psi\right)-\frac{4}{3\sqrt{\pi}}\left(\partial_t+\mathbf{v}_g\partial_y\right)\Psi^{\frac{3}{2}}\end{aligned} \tag{10}$$

where

$$\Psi=e\Phi/T_e$$

is the normalized potential and v_e^* is the electron diamagnetic drift which may be expressed as $v_e^*=\kappa T_e/eB_0$ and κ is the inverse of the scale length of the number density inhomogeneity $\kappa=-(1/n_0)/(dn_0/dx)$ and $v_{\nabla B}$ is the grad B drift given by $v_{\nabla B}=(\varrho/B_0)(c_s(dB_0/dx))$ and ϱ is the ion larmour radius given by $\varrho=c_s/\Omega_{ci}$. c_s is the ion sound velocity given as $c_s=\sqrt{T_e/m_i}$.

A similar equation can be developed for the deep well case. Here as we have $n > n_0$ and using the electron distribution given by Eq. 2, we rewrite Eq. 7 as

$$\frac{d}{dt}[\ln(\Omega_{ci}+\Omega_i)-\ln n_i]=0$$

where it may be noted that

$$\ln n_i=\ln\frac{2}{\sqrt{\pi}}n_0\sqrt{\Psi}=\ln\frac{2}{\sqrt{\pi}}+\ln n_0+\frac{1}{2}\ln\Psi$$

Thus the above expression can be recast in the form

$$\frac{d}{dt}\left[\ln\frac{\Omega_{ci}}{n_0}+\frac{\Omega_i}{\Omega_{ci}}-\frac{1}{2}\ln\Psi\right]=0$$

Finally we get the equation for deep well case as

$$\begin{aligned}&\left(\partial_t+\mathbf{v}_g\partial_y\right)\left(\varrho^2\nabla^2\Psi\right)-\left(v_e^*+v_{\nabla B}\right)\frac{\partial\Psi}{\partial y}\\&=\frac{\varrho^2}{B_0}(\nabla\Psi\times z)\cdot\nabla\left(\nabla^2\Psi\right)+\left(\partial_t+v_g\partial_y\right)\frac{1}{2}\ln\Psi\end{aligned} \tag{11}$$

where once again the higher order terms have been neglected as was done for the shallow well case.

We see here that the presence of trapped particles in both shallow and deep well cases produces a modified HM equation by the addition of another non-linear term (the second term on the right-hand sides of Eqs. 10 and 11). We note that this term makes a larger contribution than the original non-linear HM term (the first term on the right-hand side of the above equation). We may thus drop the original HM non-linear term in Eqs. 10 and 11 to obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v_{gy}\partial_y\right)\left(\varrho^2\nabla^2\Psi-\Psi\right)-\left(v_e^*+v_{\nabla B}\right)\frac{\partial\Psi}{\partial y}\\&=-\frac{4}{3\sqrt{\pi}}\left(\frac{\partial}{\partial t}+v_{gy}\partial_y\right)\Psi^{\frac{3}{2}}\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\varrho^2\left(\frac{\partial}{\partial t}+v_{gy}\partial_y\right)\nabla^2\Psi-\left(v_e^*+v_{\nabla B}\right)\frac{\partial\Psi}{\partial y}\\&=\frac{1}{2}\left(\frac{\partial}{\partial t}+v_{gy}\partial_y\right)\ln\Psi\end{aligned} \tag{13}$$

Equations 12 and 13 are the modified HM equations for the shallow and deep potential well trapped electron cases, respectively. The second term on the left-hand side of Eqs. 12 and 13 can be written as $\left(v_e^*+v_{\nabla B}\right)=\rho c_s((d/dx)\ln(B_0/n_0))$ where three distinct conditions may be considered, i.e., when $\rho c_s((d/dx)\ln(B_0/n_0))$ is >0, <0 or $=0$. The last corresponds to the frozen in condition. Different solutions of Eq. 12 are considered in the subsequent subsections. However we note here that the stationary solutions of the original HM equation (when no trapping is taken into account) are expressed through Bessel function using piecewise linear solutions [3]. This is no longer possible here and we consider certain alternate strategies to get insights to the effect of trapped electrons on the formation of vortices.

Bounce Frequencies in Potential Well

In this section we consider that the particles which become trapped in the potential well can undergo oscillatory motion in the well itself, if their energy is less than the potential energy associated with the well then these particles remain trapped. We can use the non-linear evolution

Eqs. 12 and 13. for the shallow and deep well cases to estimate the "bounce" frequency of the particles within the wells. We first consider the shallow well case and expand Ψ around fixed minimum value Ψ_0 of the potential well in the following way.

$$\Psi = \Psi_0 + \Psi_1$$

We linearize Eqs. 12 and 13 (around the value Ψ_0) and solve those Eqs. 12 and 13 by using a plane wave solution $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ to obtain

$$\omega_b = \frac{k_y\left(v_e^* + v_{\nabla B}\right)}{\left(\varrho^2k^2 + 1 - \frac{2}{\sqrt{\pi}}\sqrt{\Psi_0}\right)} \tag{14}$$

and

$$\omega_b = \frac{k_y\left(v_e^* + v_{\nabla B}\right)}{\left(\varrho^2k^2 + \frac{1}{2\Psi_0}\right)} \tag{15}$$

where $\omega_b = \omega - k_yv_{gy}$ is the bounce frequency as the particle is reflected off the walls of the potential well for the shallow and deep well cases, respectively.

We note that when we compare the terms in the denominators of Eqs. 14 and 15 and take into account the fact that in most cases $\varrho^2k^2 < 1$ then we see that for the shallow well case Eq. 14 the value of the trapping potential Ψ_0 only has a small corrective effect on the bounce frequency ω_b. On the other hand for the deep potential well, both terms in the denominator can be of the same order making the bounce frequency large. Thus we can conclude that in case of the deep well, the fixed value of the potential has a more significant effect on the bounce frequency of the trapped particles.

We see in Fig. 1 for the shallow well case that as the potential rises above 0.9, the bounce frequency becomes infinity which means that the particle no longer remains in the potential well. However in the deep well case Fig. 2 we can see that as the potential grows, the bounce frequency increases gradually and the particles remain trapped. Here we can conclude that the bounce frequency for the deep well case has a more significant effect as compared to that in the shallow well case.

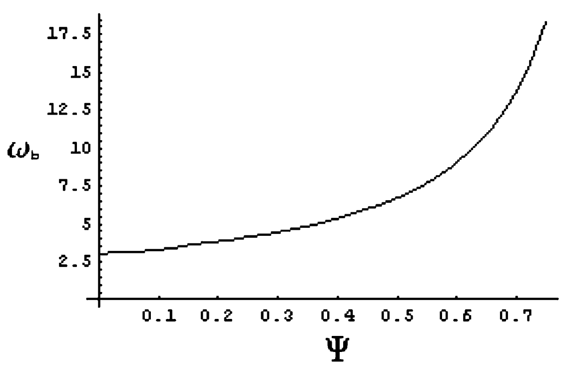

Fig. 1 Potential versus bounce frequency for shallow well

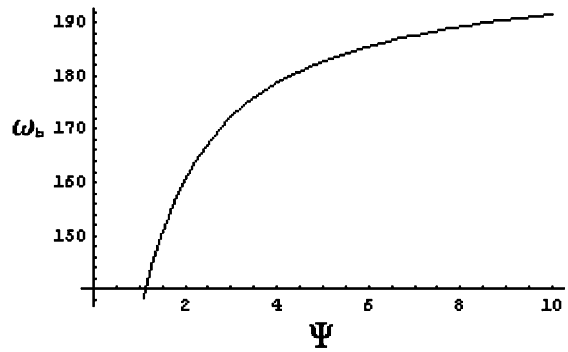

Fig. 2 Potential versus bounce frequency for deep well

Analytic Solutions for the Shallow Potential Well Case

In this section we consider only the shallow well case and derive a Kortweg deVries (KdV) type of equation from Eq. 12. For this we use the reductive perturbation (stretched variable) technique which is introduced in the following manner [25]

$$\eta = \varepsilon^{\frac{1}{4}}(y - ut), \quad \tau = \varepsilon^{\frac{3}{4}}t, \quad x = x$$

where u is the speed of the perturbation in the comoving frame of reference. The potential perturbations are expanded as follows

$$\Psi_1 = \varepsilon\Psi_1 + \varepsilon^{\frac{3}{2}}\Psi_2$$

Using these perturbations in Eq. 12, we get in the lowest order of ε (i.e. $\varepsilon^{\frac{5}{4}}$)

$$\left(v_g - u\right)\left(\varrho^2\partial_{xx} - 1\right)\partial_\eta\Psi_1 - \left(v_e^* + v_{\nabla B}\right)\partial_\eta\Psi_1 = 0 \tag{16}$$

which corresponds to the linear case. Here we introduce a separation of variables

$$\Psi_1 = A(\eta, \tau)Y(x)$$

so that from Eq. 16 we obtain

$$\ddot{Y} = \frac{1}{\varrho^2}(\upsilon + 1)Y \tag{17}$$

where $\upsilon = \left(\mathbf{v}_e^* + \mathbf{v}_{\nabla B}\right)/\left(\mathbf{v}_g - u\right)$. In the next order $(\tilde{}\varepsilon^{7/4})$ we obtain,

$$\partial_\tau\left(\varrho^2\partial_{xx} - 1\right)\Psi_1 + \left(v_g - u\right)\partial_\eta\left(\varrho^2\partial_{xx} - 1\right)\Psi_2 + \varrho^2\left(v_g - u\right)\partial_{\eta\eta\eta}\Psi_1 - \left(v_e^* + v_{\nabla B}\right)\partial_\eta\Psi_2 + \frac{4}{3\sqrt{\pi}}\left(v_g - u\right)\partial_\eta\Psi_1^{\frac{3}{2}} = 0 \tag{18}$$

Following the method outlined in Dodd et al. [25], we can write Eq. 18 as

$$\partial_\tau\left(\varrho^2\ddot{Y}-Y\right)A+\varrho^2\left(v_g-u\right)Y\partial_{\eta\eta\eta}A+\frac{4}{3\sqrt{\pi}}\left(v_g-u\right)Y^{\frac{3}{2}}\partial_\eta A^{\frac{3}{2}}=0$$

using Eq. 17, finally we get

$$\partial_t A + a\partial_{\eta\eta\eta}A + b\partial_\eta A^{\frac{3}{2}} = 0 \tag{19}$$

where a, and b are the coefficients given by respectively,

$$a = \frac{\rho^2(v_g-u)\int_{x_1}^{x_2} Ydx}{\int_{x_1}^{x_2} vYdx}$$

$$b = \frac{4}{3\sqrt{\pi}}(v_g-u)\frac{\int_{x_1}^{x_2} Y^{3/2}dx}{\int_{x_1}^{x_2} vYdx}$$

The above integrals can be solved by using appropriate boundary conditions. The solution to Eq. 19 is given as

$$\Psi = \frac{25}{16}\left(\frac{\lambda}{b}\right)^2 \sec h^4\left(\frac{1}{4}\sqrt{\frac{\lambda}{a}}(\eta-\lambda t)\right) \tag{20}$$

We note that we have used the boundary conditions that as $x \to \pm\infty$, $Y, \frac{\partial Y}{\partial x}, \ldots \to 0$. Here $\frac{25}{16}\left(\frac{\lambda}{b}\right)^2$ is the amplitude and where λ is the speed of comoving frame of reference.

Another form of the solution which can be obtained for Eq. 12 proceeds by moving to a comoving frame of reference by setting $\eta = y - ut$, then Eq. 12 is cast in the form

$$\nabla^2\Psi = \alpha\Psi + \beta\Psi^{\frac{3}{2}} \tag{21}$$

Here

$$\alpha = \frac{1}{\varrho^2}\left(1+\frac{\left(v_{\mathbf{e}}^* + v_{\nabla\mathbf{B}}\right)}{v_{\mathbf{g}}-u}\right)$$

and $\beta = -\frac{4}{3\sqrt{\pi}\varrho^2}$

Equations of this type have a general solution of the form Ref. [26]

$$\Psi = \left(\frac{-2\beta\cos h^2 z}{\alpha(n+1)}\right)^{\frac{1}{1-n}} \tag{22}$$

where $n = \frac{3}{2}$ in our case and such a solution is valid when $\alpha > 0$ and $\beta\ (n + 1)\ < 0$

In the solution given by Eq. 22

$$z = -\frac{\sqrt{a}}{4}(x\sin c_1 + y\cos c_1) + c_2$$

where c_1 and c_2 are arbitrary constants thus in general the solution to Eq. 21 is given by

$$\Psi = \frac{225\pi}{256}\left(1+\frac{(v_e^*+v_{\nabla B})}{v_g-u}\right)^2 \sec h^4\left(c_2 - \frac{1}{4\varrho}\sqrt{1+\frac{(v_e^*+v_{\nabla B})}{v_g-u}}(x\sin c_1 + y\cos c_1)\right) \tag{23}$$

We see that both solutions, i.e., those given by Eqs. 20 and 23, the two solutions are of the same form.

Sagdeev Potential

We can also investigate the solutions of the Eqs. 12 and 13 by employing the Sagdeev Potential approach [18]. Eq. 21 for the shallow well trapping is rewritten with the help of the Sagdeev (pseudo) potential V in the form

$$-\frac{dV}{d\Psi} = \alpha\Psi - \beta\Psi^{\frac{3}{2}}$$

The Sagdeev potential V can be found by integrating above equation and using the boundary condition, i.e., $\Psi = 0$ at $x = 0$.

$$V = \beta\frac{2}{5}\Psi^{\frac{5}{2}} - \frac{\alpha}{2}\Psi^2 \tag{24}$$

Similarly for the deep well case, we may write from Eq. 13

$$\frac{dV}{d\Psi} = \beta\Psi - \alpha\ln\Psi$$

Integrating the above equation w.r.t Ψ we get

$$V = \frac{\beta}{2}\Psi^2 - \alpha(\Psi\ln\Psi - \Psi) \tag{25}$$

For the shallow well case, we obtain minima for $\alpha\ < \beta$ at

$$\Psi = \left(\frac{\alpha}{\beta}\right)^2$$

For the deep well case, we obtain minima for $\alpha\ > \beta$ at

$$\Psi = \frac{\alpha}{\beta}\ln\Psi$$

This is evident from Figs. 7 and 8 also.

If it is a potential well as in our case, a particle entering from left will go to the right-hand side of the well, reflect and return to $x = 0$ making a single transit. Such a pulse is a soliton, propagating to the left with an arbitrary velocity u. Now, if a particle suffers a loss of energy while in the well, it will never return to $x = 0$ but will oscillate in time about some positive value of x. This behavior is depicted in Figs. 7 and 8.

Two-Dimensional Potential Well

In this section we discuss the same problem but now with two-dimensional trapping. We adopt the same strategy to derive a modified HM-equation but we first calculate the number density in two dimensions again by following [21]. We thus have

$$n_e = n_0 A \int_{-\infty}^{\infty} dP_{\parallel} \int_0^{\infty} dP_{\perp} \exp\left(-\frac{P_{\parallel}^2}{2mT_e} - \frac{P_{\perp}^2}{2mT_e} + \Psi\right)$$

where A is the normalization factor and $P_{\perp}, P_{\parallel}, m, T_e$ are perpendicular and parallel momentum with respect to ambient magnetic field B_0, mass and temperature of electrons, respectively. and $\Psi = \frac{e\Phi}{T_e}$ is the normalized potential. After integration with respect to $P_{\parallel}$, we obtain

$$n_e = (2\pi m T_e)^{\frac{3}{2}} n_0 \left[\int_0^{P_{\perp}} P_{\perp} dP_{\perp} + \int_{P_{\perp}}^{\infty} P_{\perp} \exp\left(-\frac{P_{\perp}^2}{2mT_e} + \Psi\right) dP_{\perp}\right]$$

from where we finally obtain

$$n_e = n_0(1 + \Psi)$$

Using this in Eq. 7, we obtain in the same way for two-dimensional trapping for the shallow and deep potential trapped electrons, respectively

$$\frac{d}{dt}\left(\varrho^2\nabla^2\Psi - \Psi\right) - \left(v_e^* + v_{\nabla B}\right)\frac{\partial\Psi}{\partial y} = \frac{\varrho^2}{B_0}(\nabla\Psi \times z)\cdot\nabla\left(\nabla^2\Psi\right) - \frac{1}{2}\frac{d\Psi^2}{dt} \tag{26}$$

and

$$\varrho^2\frac{d}{dt}\nabla^2\Psi - \left(v_e^* + v_{\nabla B}\right)\frac{\partial\Psi}{\partial y} = \frac{\varrho^2}{B_0}(\nabla\Psi \times z)\cdot\nabla\left(\nabla^2\Psi\right) + \frac{d}{dt}\ln(1+\Psi) \tag{27}$$

By considering Eq. 27 we see that we cannot ignore the non-linear HM term on the right-hand side since the order of the non-linearity due to trapping is the same as that in the original HM-equation. Thus for the shallow potential case both non-linear terms are retained. For the case of a deep well we see that if we compare Eqs. 27 and 13 it is seen that both equations, i.e., one-dimensional and two-dimensional are similar in structure. Thus we conclude here that trapping effect in the one-dimensional is the same as in the two-dimensional trapping in deep well case while for the shallow well case, the trapping effect is stronger in the one-dimensional case than in the present two-dimensional case.

Solution of Shallow Potential Well Case

We now obtain a KdV equation for the two-dimensional shallow well trapping case by introducing the stretched variables in the following manner:

$$\eta = \varepsilon^{\frac{1}{2}}(y - ut)$$
$$\tau = \varepsilon^{\frac{3}{2}}t$$
$$x = x$$

and the potential is expressed as

$$\Psi = \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \cdots.$$

Using these perturbations in Eq. 24 we get for the lowest order (i.e., $\varepsilon^{\frac{3}{2}}$)

$$(v_g - u)\partial_\eta\left(\varrho^2\partial_{xx} - 1\right)\Psi_1 - \left(v_e^* + v_{\nabla B}\right)\partial_\eta\Psi_1 = 0 \tag{28}$$

In the next order ($\varepsilon^{\frac{5}{2}}$), we obtain

$$\partial_\tau\left(\varrho^2\partial_{xx} - 1\right)\Psi_1 + (v_g - u)\partial_\eta\left(\varrho^2\partial_{xx} - 1\right)\Psi_2 + \varrho^2(v_g - u)\partial_{\eta\eta\eta}\Psi_1 - \left(v_e^* + v_{\nabla B}\right)\partial_\eta\Psi_2 = \frac{\varrho^2}{B_0}\left[\partial_\eta\Psi_1\partial_{xxx}\Psi_1 - \partial_x\Psi_1\partial_{\eta xx}\Psi_1\right] + \frac{1}{2}(v_g - u)\partial_\eta\Psi_1^2$$

Once again following [24], we get

$$\partial_t A + \alpha\partial_{\eta\eta\eta}A - \beta\partial_\eta A^2 = 0 \tag{29}$$

This is the standard KdV equation and constants α ,β are given by, respectively

$$\alpha = \frac{-\int_{-\infty}^{\infty} Y^2 dx}{\int_{-\infty}^{\infty} \frac{Y^2(\varrho^2 v + 1)}{\varrho^2(v_g - u)} dx}$$

$$\beta = \frac{\int_{-\infty}^{\infty} \frac{Y^2}{\varrho^2} dx}{\int_{-\infty}^{\infty} \frac{Y^2(\varrho^2 v + 1)}{\varrho^2(v_g - u)}} - \frac{1}{B_0}\frac{\int_{-\infty}^{\infty} \frac{vY^3}{v_g - u} dx}{\int_{-\infty}^{\infty} \frac{Y^2(\varrho^2 v + 1)}{\varrho^2(v_g - u)}}$$

where

$$v = \frac{(v_g - u + v_e^* + v_{\nabla B})}{\varrho^2(v_g - u)}.$$

The solitary vortex solution is given by

$$A = A_0 \sec h^2 \kappa\varsigma$$

where $A_0 = -v/2\beta$ is the amplitude, $\varsigma = \eta - vt$ denoting that we have a stationary solution and $\kappa = \sqrt{v/4\alpha}$ is the inverse width of the soliton.

Bounce Frequencies in Potential Well

We can use the non-linear evolution Eqs. 26 and 27 for the shallow and deep well cases to estimate the "bounce" frequency of the particles within the wells. We first consider the shallow well case and expand Ψ around the fixed value Ψ_0 of the potential well in the same way as done in section "Bounce Frequencies in Potential Well".

$$\Psi = \Psi_0 + \Psi_1$$

By taking $\Psi_1 \sim \exp i(k \cdot r - \omega t)$, we obtain the bounce frequencies for the shallow and deep well cases, respectively.

$$\omega_b = \frac{k_y\left(v_e^* + v_{\nabla B}\right)}{(\varrho^2 k^2 + 1 - 2\Psi_0)}$$

and

$$\omega_b = \frac{k_y\left(v_e^* + v_{\nabla B}\right)}{(\varrho^2 k^2 - (1 + \Psi_0))}$$

where ω_b is the bounce frequency given by $\omega_b = \omega - k_y v_{gy}$ same as in section "Bounce Frequencies in Potential Well". We see that the expressions for the bounce frequencies in both cases are similar to that of the shallow well case in the one-dimensional case.

Discussion and Conclusion

The basic model equation which is derived in section "Two-Dimensional Potential Well" is a modified HM equation. The modification occurs due to the effect of one-dimensional trapping. We see that we obtain stronger non-linear terms when trapping is taken into account as compared to the standard HM equation and it was for this reason that we ignored the non-linearities of the original HM equation which is small in comparison with the non-linear term which arises due to the trapped particle effect in case of one-dimensional problem. Further we split the problem into two cases, namely the shallow potential well case and the deep potential well case. We have shown that there exists a family of localized vortex solutions of Eq. 12 which are stationary and have strongly non-linear structures. Next we calculated the bounce frequency of the electrons within the potential well. We noted that in the deep well case, the potential due to trapping has a stronger effect on the bounce frequency than in the shallow well case.

Further in section "Analytic Solutions for the Shallow Well Case" we made use of the reductive perturbation method and reduced the modified HM equation Eq. 12 for the shallow well case and obtained a KdV equation with a non-linearity of power 3/2. We also obtained an analytic solution to this equation. In section "Analytic Solutions for the Shallow Well Case" we also gave another stationary solution to the HM equation. This is given by Eq. 23. In section "Sagdeev Potential" we analyzed Eqs. 12 and 13 for both the shallow and deep well cases, via the Sagdeev Potential approach. It was shown in both cases that solitary vortices can exist. In section "Bounce Frequencies in Potential Well" we analyzed the bounce frequencies for electrons trapped in both shallow and deep one-dimensional potential wells. It is observed that the bounce frequency in the deep well is more significant than in the shallow well.

In section "Two Dimensional Potential Well" we considered two-dimensional trapping of electrons and investigated the evolution of vortices. We saw that in the shallow well case the non-linearity which arises is of the same order as the non-linearity occurring in the original HM equation. Hence in this case both terms have to be included in the analysis and in section "Solution of Shallow Well Case" we have obtained the usual KdV equation for two-dimensional trapping in the shallow well case. However in the case of a deep potential well we observe that it does not differ from the corresponding one-dimensional case. We then investigated the bounce frequencies for the deep well and the shallow well cases and noted that the two frequencies are nearly the same and also similar to the shallow well case in one dimension. While the bounce frequency in the deep well case in one dimension has stronger influence than in any other cases. Thus we conclude here that the bounce frequency in one dimension is more significant than in two dimensions as is also evident in Figs., 1 and 2 for the shallow and deep potential well cases, respectively.

These results have been shown graphically. We have plotted KdV solution given by Eq. 20 in one-dimensional shallow well case in Figs. 3, and 4 and by giving different arbitrary values to the constants a, b, and λ, we got different values of the amplitudes of the solution. We took $a = 1$, $b = 1$, $\lambda = 1$ for Fig. 3 and $a = 1$, $b = 1$, $\lambda = 10$ for Fig. 4 in which we get different amplitudes. Then we plot the graphs for the Eq. 23 for the shallow well case in one dimensions in Figs. 5 and 6 in which amplitude is plotted against its two arguments x and y. It is noted that by giving different values to constants C_1 and C_2, we get the rotation of amplitude in perpendicular direction. In Fig. 5, we took $C_1 = \frac{\pi}{10}$, $C_2 = 0$ and in Fig. 6, $C_1 = -\frac{\pi}{4}$ and $C_2 = 0$ and noted that the amplitude is rotated.

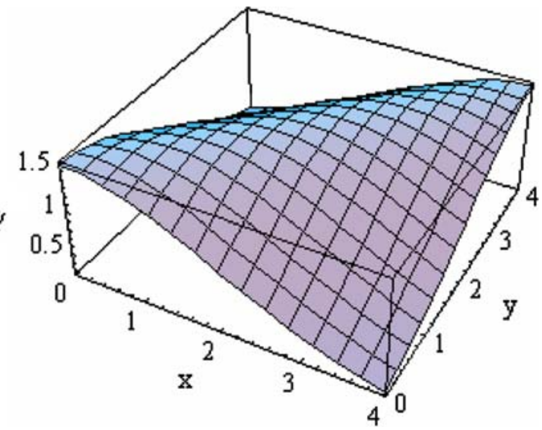

Fig. 3 KdV solution for shallow well when $a = b = 1$, $\lambda = 1$

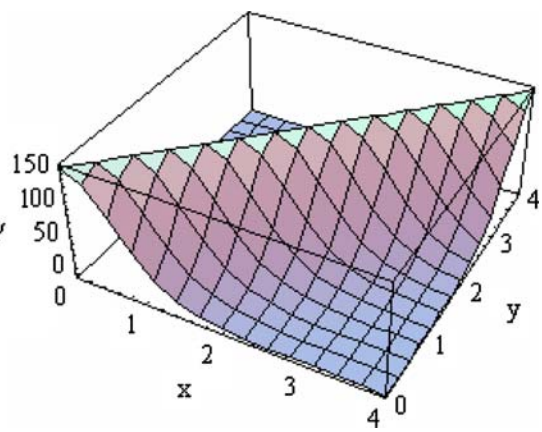

Fig. 4 KdV solution for shallow well when $a = b = 1$, $\lambda = 10$

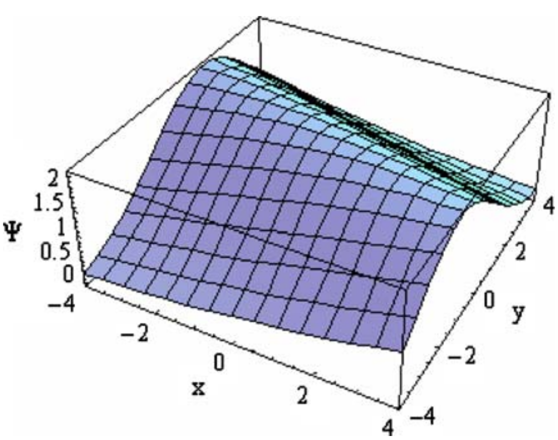

Fig. 5 Solution for shallow well when $C_1 = \frac{\pi}{10}$, $C_2 = 0$

Analysis using the Sagdeev Potential method for the shallow and deep potential wells confirmed the existence of solitary vortices for both the shallow and deep potential well cases .We also obtained conditions for the occurrence of minimas of the potential wells. These results were presented graphically in Figs. 7 and 8.

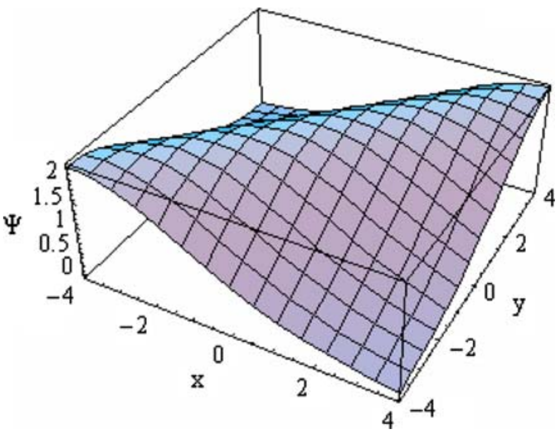

Fig. 6 Solution for shallow well when $C_1 = -\frac{\pi}{4}$, $C_2 = 0$

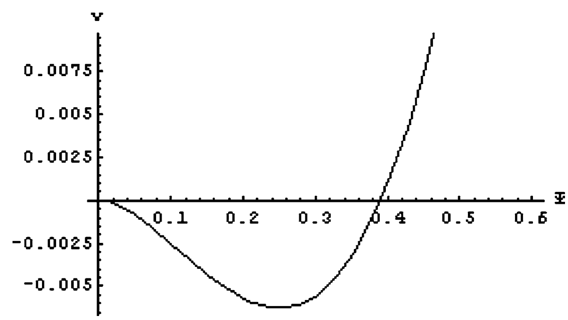

Fig. 7 Sagdeev potential for shallow well

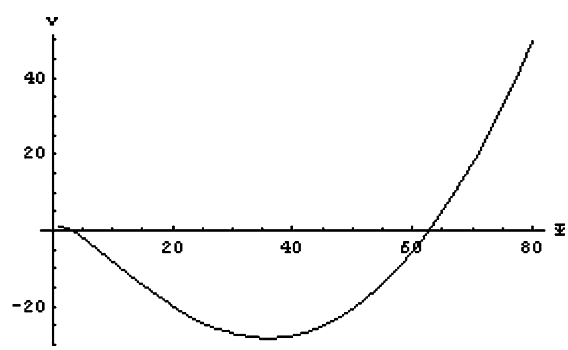

Fig. 8 Sagdeev potential for deep well

The originality of our study was to start from an exact and realistic solution of the modified Hasegawa Mima equation and to provide analytical results and relevant parameters for the trapping process. Moreover, we have considered the effect of small-scale fluctuations on the motion of the trapped electrons inside the vortex structure. These theoretical result have been formulated in the context of electron–ion plasma but they can clearly have applications in other fields of astrophysics and geophysics.

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