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Influence of non-monochromaticity on zonal-flow generation by magnetized Rossby waves in the ionospheric E-layer

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Abstract. The influence of non-monochromaticity on low-frequency, large-scale zonal-flow nonlinear generation by small-scale magnetized Rossby (MR) waves in the Earth's ionospheric E-layer is considered. The modified parametric approach is used with an arbitrary spectrum of primary modes. It is shown that the broadening of the wave packet spectrum of pump MR waves leads to a resonant interaction with a growth rate of the order of the monochromatic case. In the case when zonal-flow generation by MR modes is prohibited by the Lighthill stability criterion, the so-called two-stream-like mechanism for the generation of sheared zonal flows by finite-amplitude MR waves in the ionospheric E-layer is possible. The growth rates of zonal-flow instabilities and the conditions for driving them are determined. The present theory can be used for the interpretation of the observations of Rossby-type waves in the Earth's ionosphere and in laboratory experiments.

1. Introduction

The existence of anisotropic large-scale structures, such as convective cells, zonal flows and jets has been intensively investigated both in laboratory plasmas (Shukla et al. 1981, 1984, 2002; Terry 2000; Diamond et al. 2005) and in geophysical fluid dynamics (Busse 1994; Rhines 1994). Both ground-based and satellite observations clearly show that, for different layers of the ionosphere, there are sheared zonal flows. These are associated with azimuthally symmetric band-like flows propagating along the parallels with inhomogeneous velocities along the meridians (see, e.g., Petviashvili and Pokhotelov 1992). At the same time a large amount of observational data verify the permanent existence of ultra-low-frequency (ULF) planetary-scale perturbations in the E- and F-regions of the ionosphere (see, e.g., Lawrence and Jarvis 2003). Among them, special attention must be paid to Rossby-type perturbations propagating at a fixed latitude along the parallels around the Earth. In reality, the Earth's ionospheres can support both propagating waves and zonal flows and they thus constitute dynamical systems which exhibit complex nonlinear interactions. In this way, the ionospheric medium builds up conditions which are also favorable to the formation of electromagnetic nonlinear stationary solitary wave structures (Pokhotelov et al. 1996, 2001; Kaladze et al. 2003).

The idea of generation of zonal flows by tropospheric Rossby waves on the basis of the kinetic equation for wave packets was put forward by Smolyakov et al. (2000). Using the formulation of parametric instabilities on the basis of a three-wave resonant nonlinear interaction, the theory of zonal-flow generation by Rossby waves was further developed by Shukla and Stenflo (2003) and Onishchenko et al. (2004). In these papers it was shown that zonal flows in a non-uniform rotating neutral atmosphere can be excited by finite-amplitude Rossby waves. Accordingly, these papers study the interaction of pump waves (Rossby waves), a sheared flow and two satellites of the pump wave (side-band waves). This approach is an alternative to the standard weak turbulence approach used by Smolyakov et al. (2000). The driving mechanism of this instability is due to the Reynolds stresses, which are inevitably inherent for finite-amplitude small-scale Rossby waves. Owing to this essential nonlinear mechanism, spectral energy transfers from small-scale Rossby waves to large-scale enhanced zonal flows (inverse cascade) in the Earth's neutral atmosphere. In addition, the zonal-flow generation was considered within a simple model of Rossby wave turbulence, using the classical nonlinear two-dimensional Charney equation. It was found that the necessary condition for zonal-flow generation is similar to the Lighthill criterion for modulation instability in nonlinear optics (Lighthill 1965).

The question arises: are there other zonal-flow generation mechanisms? To this end Kaladze et al. (2007a) added a scalar nonlinearity of Korteweg–de Vries type to the Charney equation. It was shown that in this case zonal-flow generation by the Rossby waves always exists and needs no criterion fulfilment.

The problem of zonal-flow generation by a monochromatic Rossby wave packet was considered in Kaladze et al. (2007a), Onishchenko et al. (2004) and Shukla and Stenflo (2003). The idea of investigating the influence of non-monochromaticity of wave packet in general was put forward by Smolyakov et al. (2000) and Malkov et al. (2001). Smolyakov et al. (2000) showed that the broadening of the wave packet gives the possibility of considering zonal-flow generation in the 'hydrodynamic' and 'kinetic' regimes, similar to the case of plasma beam instabilities. Note that in the kinetic regime resonant-type instability takes place, whereas the hydrodynamic regime refers to a coherent instability.

An essential contribution to the problem of zonal-flow generation by primary (pump) modes with arbitrary spectrum broadening was made by Mikhailovskii et al. (2006b), where the modified parametric approach was suggested. In this approach, the driving forces of zonal flows are represented by a summation (or integration) over the spectrum of the primary modes, which gives the possibility of revealing additional zonal-flow generation mechanisms. The idea of using two Gaussian wave packets in the problem of zonal-flow generation comes from Malkov et al. (2001). This was used by Mikhailovskii et al. (2006a) to show a new mechanism of zonal-flow generation by Rossby waves in a shallow rotating neutral fluid, called the two-stream-like instability. Unfortunately, no expression for the growth rate concerning the excitation of zonal flow by ordinary Rossby waves was obtained in Mikhailovskii et al. (2006a) and there does not appear to be any discussion of this question in that paper.

In the present paper we will focus our attention on the Earth's weakly ionized, incompressible ionospheric gas of the E-layer ($\approx$ 90–150 km from the Earth's surface). Unlike the neutral atmosphere, such a gas becomes conductive and the influence of electromagnetic forces should be taken into account. In such an ionospheric gas, MR waves can propagate (see Sec. 2). The problem of zonal-flow excitation by MR waves in the ionospheric E-layer by a monochromatic wave packet was initiated by Kaladze et al. (2007a), where it was shown that the zonal-flow instability by these monochromatic waves is also prohibited when the Lighthill instability criterion is not fulfilled. However, no investigations has been carried out so far into the influence of non-monochromaticity of wave packets on zonal-flow generation by MR waves or to identify any additional mechanisms of instability.

In the present paper the problem of zonal-flow generation by small-scale ($\lambda < r_{\mathrm{R}}$, where r_{R} is modified by the inhomogeneous geomagnetic field Rossby radius) MR waves in the ionospheric E-layer is further developed taking into account the broadening of the wave packet spectrum. To this end we examine the problem by considering primary MR waves having a sufficiently broad-spectrum wave packet and will show that a two-stream-like instability is also an effective mechanism for the excitation of a zonal-flow instability.

In Sec. 2 a brief description of the MR waves propagating in the ionospheric E-layer is presented. Owing to the assumption of a distinct time- and space-scale separation between the turbulent oscillations and the zonal flow on the basis of the nonlinear Charney equation, the zonal-flow dispersion equation is derived in Sec. 3. To this end, by analogy with Kaladze et al. (2007a), the spectral analysis of the problem is carried out considering the a three-wave resonant parametric interaction between the zonal flows, the primary (pump) MR modes generating the flows, and the side-band amplitudes, which depend on both the primary modes and zonal flows. Then, following Mikhailovskii et al. (2006b), the driving force of zonal flow is calculated, which is in fact the Reynolds stress, expressed as a summation over contributions of the primary modes of the wave packet. The results of Kaladze et al. (2007a) for a small-scale ($\lambda < r_{\mathrm{R}}$) monochromatic wave packet of primary MR modes are obtained as a limiting case. The determination of their growth rates is useful for the generalization of zonal-flow generation by non-monochromatic wave packets of primary modes. The influence of non-monochromaticity effects on zonal-flow generation under Lighthill's instability criterion is investigated in Sec. 4. The case of zonal-flow generation by 'single-humped' Gaussian wave packets is considered. Limiting cases for a sufficiently small spectrum broadening and a strong broadening are distinguished. In Sec. 5 'two-humped' spectra of the primary MR wave are investigated to show the existence of a new, two-stream-like mechanism of zonal-flow generation under the conditions when the Lighthill instability criterion is not satisfied. The main results of the paper are discussed in Sec. 6.

2. Magnetized Rossby waves in the E-layer

In contrast to the neutral atmosphere the ionospheric E-layer consists of neutral and charged particles, which makes the ionosphere conductive. Therefore, the interaction of inductive currents with the geomagnetic field should be taken into account. It was recently shown (Kaladze and Tsamalashvili 1997; Kaladze 1998, 1999; Kaladze et al. 2003, 2004) that MR waves can propagate in the ionospheric E-layer.

The term 'magnetized Rossby waves' was introduced by Kaladze (1999) for the ionospheric generalization of tropospheric Rossby waves in a rotating atmosphere by the spatially inhomogeneous geomagnetic field. The theory of MR waves for linear and nonlinear stages was developed by Kaladze et al. (2004). The MR waves belong to the ULF range $(10^{-5}$–$10^{-4})$ s^{-1}, with wavelength of the order 1000 km and longer, and the phase velocity is of the order the velocity of the local winds, i.e. $\sim$(1–100) m s^{-1}. MR waves do not significantly perturb the geomagnetic field. For the typical ionization fraction in the E-layer, the Lorentz force is comparable to the Coriolis force, both having a spatial inhomogeneity scale of the order of Earth's radius. Thus, they are induced by the latitudinal inhomogeneity both of the Earth's angular velocity and of the geomagnetic field given later by β and α, respectively. The Lorentz force opposes the Coriolis force vorticity and therefore partial or full compensation of the Coriolis deviation by the magnetic deviation is possible. Correspondingly, the propagation phase velocity of the linear waves also decreases. The MR waves are excited solely by the ionospheric dynamo electric field when the Hall effect due to the interaction with the ionized ionospheric component in the E-layer is included.

In the following analysis the local Cartesian coordinate system (x, y, z) is used with the x-axis directed to the east (longitudinal direction), the y-axis directed to the north (latitudinal direction) and the z-axis coinciding with the local vertical direction.

Short-scale MR waves are propagating in the middle-latitude E-layer of the ionosphere and their dispersion may be written as follows (Kaladze et al. 2004):

$$\omega_{\mathbf{k}} = -\frac{k_x(\alpha+\beta)}{k_\perp^2}. \tag{1}$$

Here $\omega_{\mathbf{k}}$ is the wave frequency, $\mathbf{k}$ is the wave vector, $k_\perp = (k_x^2 + k_y^2)^{1/2}$, and $k_{x,y}$ is the x, y component of the wave vector. The value of $(\alpha + \beta)$ in (1) represents the generalized Rossby parameter, where

$$\alpha = \frac{\partial}{\partial y}\left(\frac{enB_{0z}}{\rho}\right), \quad \beta = \frac{\partial}{\partial y}(2\Omega_{0z}). \tag{2}$$

Here B_{0z} and Ω_{0z} are the z-components of the dipole geomagnetic field and the angular velocity of the Earth's rotation, respectively. In (1) the quantities α and β are related to the latitude $\lambda = \lambda_0$, e is the magnitude of the electron charge, n is the equilibrium number density for the charged particles, $\rho = Nm$ is the neutral gas mass density, and N is the equilibrium number density of neutrals. The dynamics of propagation essentially depend on the generalized Rossby parameter $(\alpha+\beta)$ and the modified Rossby radius. The parameters α and β are comparable in magnitude ($\beta \simeq -\alpha \simeq 10^{-11}$ m^{-1} s^{-1}) for the ionospheric E-layer and α depends on the ratio n/N of the charged particle number density to the neutral density. This ionization fraction is distinctly different for the night side and day side of the Earth and consequently the quantity $(\alpha + \beta)$ in (1) may change its sign. Thus, unlike Rossby waves in a neutral atmosphere, the MR waves in the ionospheric E-layer can propagate westward or eastward at a fixed latitude along parallels. They are weakly damped.

3. Nonlinear interaction of magnetized Rossby waves and zonal flow in the E-layer

The shallow rotating water model needs the existence of a free surface, which is difficult to justify in the case of the ionospheric E-layer. We prefer, therefore, to use the momentum equation for the neutral wind (Kaladze and Tsamalashvili 1997; Kaladze et al. 2007b), and will consider the nonlinear dynamics of zonal flows and their interactions with small-scale MR wave turbulence in the scope of the simple but illuminating Charney model. In this case of small-scale turbulence, when $kr_{\mathrm{R}} \gg 1$, only vectorial nonlinearity is responsible for the parametric instability. The Charney equation for MR waves in the ionospheric E-layer has the following form (Kaladze and Tsamalashvili 1997; Kaladze et al. 2007b):

$$\frac{\partial \Delta\varphi}{\partial t} + \frac{\partial \varphi}{\partial x}(\alpha + \beta) + J(\varphi, \Delta\varphi) = 0, \tag{3}$$

where the Poisson bracket operator $J(a, b) = \partial_x a \partial_y b - \partial_y a \partial_x b$ represents the vector nonlinearity. Equation (3) describes the dynamics of $\Delta\varphi$, the vorticity evolution of a system; here φ is the stream function.

Since the zonal flow varies on a much longer time scale than the comparatively small-scale MR waves, one can use a multiple-scale expansion, assuming that there is a sufficient spectral gap separating the large- and small-scale motions. Following the standard procedure to describe the evolution of the coupled system (MR waves plus zonal flows), we split the perturbation of the stream function in (3) into three components (Onishchenko et al. 2004; Mikhailovskii et al. 2006a, 2006b; Kaladze et al. 2007a):

$$\varphi = \tilde{\varphi} + \hat{\varphi} + \bar{\varphi}, \tag{4}$$

where

$$\tilde{\varphi} = \sum_{\mathbf{k}} [\tilde{\varphi}_+(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t) + \tilde{\varphi}_-(\mathbf{k}) \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_{\mathbf{k}} t)] \tag{5}$$

describes the spectrum of pump magnetized Rossby modes ($\tilde{\varphi}_-(\mathbf{k}) = \tilde{\varphi}_+^*(\mathbf{k})$, where $*$ means the complex conjugative),

$$\hat{\varphi} = \sum_{\mathbf{k}} [\hat{\varphi_+}(\mathbf{k}) \exp(i\mathbf{k}_+ \cdot \mathbf{r} - i\omega_+ t) + \hat{\varphi_-}(\mathbf{k}) \exp(i\mathbf{k}_- \cdot \mathbf{r} - i\omega_- t) + \text{c.c.}] \tag{6}$$

describes the spectrum of side-band modes, and

$$\bar{\varphi} = \bar{\varphi_0} \exp(-i\Omega t + iq_y y) + \text{c.c.} \tag{7}$$

describes the large-scale zonal-flow modes varying only along meridians; c.c. stands for the complex conjugate. The energy and momentum conservations $\omega_\pm = \Omega \pm \omega_{\mathbf{k}}$ and $\mathbf{k}_\pm = q_y \mathbf{e}_y \pm \mathbf{k}$ are fulfilled, and the pairs $(\omega_{\mathbf{k}}, \mathbf{k})$ and $(\Omega, q_y \mathbf{e}_y)$ represent the frequency and wave vector of the magnetized Rossby pump and zonal-flow modes, respectively. Considering the local approximation the amplitude $\bar{\varphi_0}$ of the zonal-flow mode is assumed to be constant.

We use the standard quasi-linear procedure, and substitute (4)–(7) into (3). Neglecting the contribution of the small nonlinear terms in the relations for the high-frequency primary modes we get

$$\frac{\partial \Delta\tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial x}(\alpha + \beta) = 0, \tag{8}$$

from which one may obtain the dispersion relation given by (1) for the MR waves.

The contribution of small nonlinear terms is essential in the case of the low-frequency zonal-flow modes. Substituting (4)–(7) into (3) and averaging out over the fast small-scale fluctuations we obtain the following evolution equation of zonal flow:

$$i\Omega\bar{\varphi}_0 = \left\langle \frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y} \right\rangle = \left\langle \frac{\partial \tilde{\varphi}}{\partial x}\frac{\partial \hat{\varphi}}{\partial y} + \frac{\partial \tilde{\varphi}}{\partial y}\frac{\partial \hat{\varphi}}{\partial x} \right\rangle$$

$$= \sum_{\mathbf{k}} k_x [2k_y(\hat{\varphi_+}\tilde{\varphi}_- + \hat{\varphi_-}\tilde{\varphi}_+) + q_y(\hat{\varphi_+}\tilde{\varphi}_- - \hat{\varphi_-}\tilde{\varphi}_+)], \tag{9}$$

where the angular brackets denote the averaging over fast oscillations. In (9) the term on the right-hand side describes the Reynolds stresses induced by the short-scale MR waves.

In order to calculate the Reynolds stresses in (9), we should find the side-band amplitudes $\hat{\varphi_\pm}$. Turning to (3), we find the equation

$$\frac{\partial \Delta\hat{\varphi}}{\partial t} + \frac{\partial \hat{\varphi}}{\partial x}(\alpha + \beta) + J(\tilde{\varphi}, \Delta\bar{\varphi}) + J(\bar{\varphi}, \Delta\tilde{\varphi}) = 0, \tag{10}$$

and for side-band amplitudes we get, respectively,

$$\hat{\varphi_\pm} = \mp i \frac{k_\perp^2}{k_{\perp\pm}^2}\frac{k_x q_y}{D_\pm}\bar{\varphi}_0\tilde{\varphi}_\pm. \tag{11}$$

We consider Ω and q_y to be small parameters and have neglected q_y^2 in comparison with $k_\perp^2$ in (11). This means that the typical scales of the zonal flows are much larger than the scales of the MR waves. In (11)

$$D_\pm = \omega_\pm \pm (\alpha + \beta)\frac{k_x}{k_{\perp\pm}^2}, \tag{12}$$

and

$$k_{\perp\pm}^2 = k_x^2 + (q_y \pm k_y)^2. \tag{13}$$

Substituting (11) into (9) and making all necessary calculations given in Kaladze et al. (2007a) and Onishchenko et al. (2004), we arrive at the following zonal-flow dispersion equation:

$$1 - \sum_{\mathbf{k}} \frac{F(\mathbf{k})}{(\Omega - q_y V_{\mathrm{g}})^2} = 0, \tag{14}$$

where

$$F(\mathbf{k}) = \frac{q_y^2 k_x^2 k_\perp^2 V_{\mathrm{g}}'}{\omega_{\mathbf{k}}}|\tilde{\varphi}_+|^2 = \frac{q_y^2 k_x^2 k_\perp^2 V_{\mathrm{g}}'}{2\omega_{\mathbf{k}}} I_{\mathbf{k}}, \tag{15}$$

with

$$I_{\mathbf{k}} = 2|\tilde{\varphi_+}|^2. \tag{16}$$

Here, $V_{\mathrm{g}}(\mathbf{k})$ is the latitudinal (y-component) pump magnetized Rossby group velocity defined by

$$V_{\mathrm{g}}(\mathbf{k}) = \frac{\partial \omega_{\mathbf{k}}}{\partial k_y} = 2\frac{k_x k_y(\alpha + \beta)}{k_\perp^4} = -2\omega_{\mathbf{k}}\frac{k_y}{k_\perp^2}, \tag{17}$$

and $V_{\rm g}' \equiv \partial V_{\rm g}/\partial k_y$ is its derivative, so that

$$V_{\rm g}' = \frac{\partial^2 \omega_{\mathbf{k}}}{\partial k_y^2} = 2k_x(\alpha+\beta)\frac{k_\perp^2 - 4k_y^2}{k_\perp^6} = -2\omega_{\mathbf{k}}\frac{k_\perp^2 - 4k_y^2}{k_\perp^4}. \tag{18}$$

Note that both $V_{\rm g}$ and $V_{\rm g}'$ can change sign owing to $\omega_{\mathbf{k}}$ (see (1)) or when $k_x = \pm\sqrt{3}k_y$.

Equation (14) is the zonal-flow dispersion relation, which is the generalization to the case of wave packets of the arbitrary spectrum of the primary MR waves (Mikhailovskii et al. 2006a,b). Thus it is possible to investigate different types of zonal-flow excitation mechanisms. It should be noted that dispersion (14) coincides in structure with (19) given by Mikhailovskii et al. (2006a), but they have mistaken the sign before summation over $\mathbf{k}$.

In the case of a monochromatic wave packet one has $F(\mathbf{k}) \sim \delta(\mathbf{k}-\mathbf{k}_0)$, and (14) reduces to (Onishchenko et al. 2004; Kaladze et al. 2007a) a hydrodynamic-type coherent instability

$$(\Omega - q_y V_{g0})^2 = -\Omega_0^2, \tag{19}$$

where

$$\Omega_0^2 = \frac{q_y^2 k_{x0}^2 k_{\perp 0}^2 |V_{g0}'|}{2|\omega_{\mathbf{k}_0}|} I_{\mathbf{k}_0}, \tag{20}$$

and the subscript ‘0’ means that appropriate values are taken at $\mathbf{k}_0$, the wave vector of this mode. It is assumed that the necessary instability condition

$$\frac{V_{g0}'}{\omega_{\mathbf{k}_0}} < 0 \tag{21}$$

is fulfilled for not too large k_{y0} (see (18)),

$$k_{y0}^2 < \tfrac{1}{3}k_{x0}^2. \tag{22}$$

Note, that the condition in (21) is the same as the Lighthill criterion for modulation instability in nonlinear optics (Lighthill 1965).

As found by Kaladze et al. (2007a), the maximum growth rate is achieved at $k_y = 0$, when (see (20))

$$\Omega_0^2 = 2q_y^2 k_{x0}^2 |\tilde{\varphi}_+|^2. \tag{23}$$

The obtained result is the standard mechanism of zonal-flow generation similar to plasma beam instability.

4. The case of a single-humped wave packet

Following Mikhailovskii et al. (2006b), we consider now the effects of non-monochromaticity of wave packets on the generation of zonal flows by MR waves. It should be noted that we will investigate in this section the zonal-flow instability mechanism, which is provided by the realization of the Lighthill instability criterion given by (21).

Consider a single non-monochromatic packet of MR waves taking the spectrum of $I_{\mathbf{k}}$ in the Gaussian form

$$I_{\mathbf{k}} = \frac{1}{\pi^{1/2}\Delta k_y}\exp\left(-\frac{(k_y - k_{y0})^2}{(\Delta k_y)^2}\right) I_{\mathbf{k}_0}. \tag{24}$$

Here k_{y0} is the centered latitudinal wave vector of the wave packet, and $\Delta k_y > 0$ is the characteristic wave packet width. The component of the wave vector k_x is assumed to be the same for all modes of the wave packet, $k_x = k_{x0}$. The summation over $\mathbf{k}$ in (14) is now understood as the integrals over k_y, and we consider the primary mode frequency $\omega = \omega_{\mathbf{k}}$ and the zonal-flow group velocity $V_{\rm g}$ as functions of k_y, i.e. $\omega = \omega(k_y)$, $V_{\rm g} = V_{\rm g}(k_y)$. Then for finite values of $\Delta k_y / k_{y0}$ the denominator in (14) becomes a function of the variable k_y and the zonal-flow dispersion relation (19) obtained in the case of a monochromatic wave packet is not valid.

Thus, instead of (14), we will use for finite $\Delta k_y / k_{y0}$ the following generalized dispersion equation:

$$1 - F(\mathbf{k}_0)\left\langle \frac{1}{(\Omega - q_y V_{\rm g})^2} \right\rangle_{k_y} = 0, \tag{25}$$

where the 'resonant denominator' $(\Omega - q_y V_{\rm g})^{-2}$ is modified by the non-monochromaticity of wave packets. Here

$$\langle (\ldots) \rangle_{k_y} = \frac{1}{\pi^{1/2} \Delta k_y} \int (\ldots) \exp\left(- \frac{(k_y - k_{y0})^2}{(\Delta k_y)^2} \right) dk_y. \tag{26}$$

4.1. Small wave packet broadening

Here we consider the case when the broadening of the wave packet is sufficiently small, i.e. $\Delta k_y / k_{y0} \ll 1$. Expanding the latitudinal group velocity $V_{\rm g}$ in series in the vicinity of k_{y0}, we obtain (Malkov et al. 2001)

$$V_{\rm g} = V_{g0} + V'_{g0}(k_y - k_{y0}). \tag{27}$$

Here the subscript '0' denotes that the corresponding function is taken for $k_y = k_{y0}$ and the prime is the derivative with respect to k_y. Note, that (76) of Mikhailovskii et al. (2006b) Δk_y is the same as $(k_y - k_{y0})$ in the expression above. In the case of the small broadening of the wave packet, integrating over k_y, we find

$$\left\langle \frac{1}{(\Omega - q_y V_{\rm g})^2} \right\rangle_{k_y} = \frac{1}{\widehat{\Omega}^2} \left(1 + \frac{3}{2} \frac{q_y^2 V_{g0}'^2}{\widehat{\Omega}^2} (\Delta k_y)^2 \right), \tag{28}$$

where

$$\widehat{\Omega} \equiv \Omega - q_y V_{g0}. \tag{29}$$

Then, instead of (19), we arrive at the following zonal-flow dispersion relation:

$$\widehat{\Omega}^2 = -\Omega_0^2 \left(1 - \frac{3}{2} \frac{q_y^2 V_{g0}'^2}{\Omega_0^2} (\Delta k_y)^2 \right). \tag{30}$$

Treating the second term in the large parentheses of (30) as a small correction, one can see that this correction leads to a decrease in the growth rate of hydrodynamic instability given by (19). Hence it can be seen that the spectrum broadening can be neglected only if

$$\Delta k_y \leqslant \Omega_0 / q_y V'_{g0}. \tag{31}$$

4.2. Arbitrary wave packet broadening

In the case where the broadening of the wave packet Δk_y is arbitrary, zonal-flow instability has a resonant character. Indeed, instead of (28) we have

$$\left\langle \frac{1}{(\Omega - q_y V_{\rm g})^2} \right\rangle_{k_y} = \frac{1}{|q_y V'_{g0}|\Delta k_y} \frac{\partial}{\partial \widehat{\Omega}} Z\left(\frac{\widehat{\Omega}}{|q_y V'_{g0}|\Delta k_y} \right), \tag{32}$$

where

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{dt \exp(-t^2)}{t - z} \tag{33}$$

is the plasma dispersion function defined for Im $z > 0$.

Then the dispersion relation given by (25) is replaced by

$$1 = -\frac{\Omega_0^2}{|q_y V'_{g0}|\Delta k_y} \frac{\partial}{\partial \widehat{\Omega}} Z\left(\frac{\widehat{\Omega}}{|q_y V'_{g0}|\Delta k_y} \right). \tag{34}$$

For $\widehat{\Omega} \ll |q_y V'_{g0}|\Delta k_y$ we obtain from (32)

$$\left\langle \frac{1}{(\Omega - q_y V_{\rm g})^2} \right\rangle_{k_y} = -\frac{2}{(q_y V'_{g0} \Delta k_y)^2} \left(1 + i\sqrt{\pi} \frac{\widehat{\Omega}}{|q_y V'_{g0}|\Delta k_y} \right). \tag{35}$$

As a result, we arrive at the zonal-flow dispersion relation

$$\widehat{\Omega} = i \frac{|q_y V'_{g0}|\Delta k_y}{\sqrt{\pi}} \left(1 - \frac{(q_y V'_{g0} \Delta k_y)^2}{2\Omega_0^2} \right). \tag{36}$$

This equation describes a kinetic zonal-flow instability. We get the instability condition

$$\Omega_0^2 > \tfrac{1}{2}(q_y V'_{g0} \Delta k_y)^2, \tag{37}$$

where Ω_0 is defined by (23). Qualitatively, this condition has the same meaning as (31).

The growth rate obtained from (36) has a maximum when the spectral broadening

$$\Delta k_y = \left(\frac{2}{3} \right)^{1/2} \frac{|\Omega_0|}{|q_y V'_{g0}|}, \tag{38}$$

and in order of magnitude is equal to

$$\gamma \simeq |\Omega_0| \sim |q_y k_{x0} \tilde{\varphi}_+|. \tag{39}$$

Note, that the role of this resonance interaction was not correctly estimated by Mikhailovskii et al. (2006b), giving the conclusion that strong broadening of the wave packet suppresses the generation of zonal flow.

5. The case of a two-humped wave packet

In this section we assume that Lighthill's instability criterion (21) is not fulfilled. It is then clear that the results obtained in Secs 3 and 4 are not representative and the system becomes stable. Meanwhile, if, instead of 'the single beam' case investigated in the previous sections, we use the two-humped wave packet distribution

$$F(\mathbf{k}) = F(\mathbf{k}_1)\delta_{\mathbf{k}\mathbf{k}_1} + F(\mathbf{k}_2)\delta_{\mathbf{k}\mathbf{k}_2}, \tag{40}$$

suggested by Malkov et al. (2001), one may obtain a two-stream-like mechanism of zonal-flow instability considered by Mikhailovskii et al. (2006a), for ordinary Rossby waves in the scope of a shallow water model. Indeed, in this case, instead of (14), one has the following dispersion relation

$$1-\frac{\Omega_1^2}{(\Omega-q_yV_{g1})^2}-\frac{\Omega_2^2}{(\Omega-q_yV_{g2})^2}=0, \tag{41}$$

where $(\Omega_1^2,\Omega_2^2)=[F(\mathbf{k}_1),F(\mathbf{k}_2)], V_{gi}\equiv V_{\mathrm{g}}(\mathbf{k}_i)$, and $i=1,2$. It is clear that when $V_{g1}=V_{g2}$ (41) has no complex roots. Thus, all growth rates should be proportional to the difference $V_{g1}-V_{g2}$.

Unlike Mikhailovskii et al. (2007) we will consider two 'strong' wave packets of non-equal intensity, which is similar to the system of two beams with non-equal densities. Consider $\Omega\sim q_yV_{g1}\sim q_yV_{g2}$, we can neglect 1 in (41) and then the solution obtained is

$$\Omega=\frac{q_y(\Omega_1^2V_{g2}+\Omega_2^2V_{g1})\pm i|q_y\Omega_1\Omega_2(V_{g2}-V_{g1})|}{\Omega_1^2+\Omega_2^2}. \tag{42}$$

The above root is valid for not too large q_y, i.e. when

$$q_y^2<\frac{\Omega_{1,2}^2}{V_{g1,2}^2}, \tag{43}$$

and the corresponding growth rate is given by

$$\operatorname{Im}\Omega=\frac{|q_y\Omega_1\Omega_2(V_{g2}-V_{g1})|}{\Omega_1^2+\Omega_2^2}. \tag{44}$$

When $V_{g2}=-V_{g1}\equiv V_{\mathrm{g}}$, and $\Omega_2^2=\Omega_1^2$, one of the four roots of (41) is purely imaginary with

$$\operatorname{Im}\Omega=|q_yV_{\mathrm{g}}|. \tag{45}$$

This solution is also valid when the inequality given by (43) is satisfied. The maximal growth rate is attained for $q_y\sim\Omega_1/V_{g1}$, and is

$$\operatorname{Im}\Omega\sim\Omega_1\sim\Omega_2. \tag{46}$$

The roots obtained here describe the two-stream-like generation of zonal flows by MR waves.

6. Conclusions

In the present study we have investigated the influence of non-monochromaticity on low-frequency, large-scale zonal-flow nonlinear generation by small-scale ($kr_{\mathrm{R}}\gg 1$) MR waves in the Earth's ionospheric E-layer. The modified parametric approach is used considering the arbitrary spectrum of primary modes. Accordingly, we study the interaction of a pump waves (Rossby waves), two satellites of the pump waves (side-band waves) and a sheared zonal flow. Thus the driving forces (the so-called Reynolds stresses) in the equation governing the evolution of zonal flows are represented as a summation (or integration) over the spectrum of the primary modes (see (9)). We have made such a generalization and thereby obtain the zonal-flow dispersion relation given by (14) for an arbitrary spectrum of the MR waves.

According to our investigation (see (14) and (15)) the possibility of zonal-flow generation by MR modes in the ionospheric E-layer is rigidly connected with the sign of $\partial^2\omega/\partial k_y^2$ for these modes (the derivative of the group velocity of the primary modes). Under the Lighthill instability condition given by (21) we have analyzed the case of zonal-flow generation by Gaussian wave packets (see (24)). According to this analysis, it seems reasonable to distinguish the limiting cases of a sufficiently small spectrum broadening, describing it in terms of an addition to the monochromatic resonant denominator (see (28)) and strong broadening, when the spectrum spread is larger than the denominator (see (35)). The maximum growth rate of such a generation is reached in both cases, i.e. of monochromatic and non-monochromatic wave packets and is expressed by (39). In the case of a monochromatic wave packet one may consider the instability to be the hydrodynamic type (see (19)), which is similar to that studied by Lawrence and Jarvis (2003) for the case of drift monochromatic wave packet with the growth rate proportional to q_y (see (39)). The broadening of the wave packet can be neglected for the condition given by (31), but even small broadening causes a decrease in its growth rate (see (30)). By increasing the broadening the instability transits to the resonant one described by (36). The wave numbers of unstable modes form a band of width given by (38), which in turn gives the maximum growth rate of the order of the hydrodynamic one (see (39)).

In the case when the zonal-flow generation by MR modes is prohibited by the Lighthill stability criterion (inverse inequality of (21)), the investigation should be continued by the elucidation of different types of zonal-flow instabilities. To this end we examined the more complicated situation of a two-humped wave packet of MR waves, which in the simplest case can be realized as two pump waves. Considering the more general case of two wave packets of non-equal intensity (see (44), (46)) we have obtained a new class of instability to add to the two-stream-like instability results of zonal-flow generation by MR waves in the Earth's ionosphere.

As is seen from our results the maximum growth rate of the zonal-flow generation is of the order of (see (39))

$$\gamma \simeq |\Omega_0| \sim |q_y k_{x0} r_{\mathrm{R}}^3 (\alpha + \beta)\tilde{\varphi}_+|. \tag{47}$$

Here the stream function $\tilde{\varphi}_+$ of pump modes is normalized by $v_{\mathrm{R}} r_{\mathrm{R}}$, where

$$v_{\mathrm{R}} = -(\alpha + \beta) r_{\mathrm{R}}^2 \tag{48}$$

is the modified Rossby velocity and r_{R} is the modified Rossby radius, respectively (Kaladze et al. 2004). For the regime considered here $q_y r_{\mathrm{R}} \sim 1, k_{x0} r_{\mathrm{R}} \gg 1$ and for typical parameters of the ionosphere $(\alpha + \beta) \approx 10^{-11}$ m^{-1} s^{-1}, $k_{x0} r_{\mathrm{R}} \simeq 10$, $\tilde{\varphi}_+ \approx 10^{-2}$, $r_{\mathrm{R}} \approx 10^6$ m, we obtain $\gamma \approx 10^{-6}$ s^{-1}. This estimate is consistent with existing observations and our investigation provides an essential nonlinear mechanism for the transfer of spectral energy from short-scale MR waves to long-scale enhanced zonal flows in the Earth's ionosphere. The peculiar feature of this instability is that it appears solely for MR waves that are localized in the cone bounded by the caustics for which $V_g'/\omega_{\mathbf{k}} = 0$. This can lead to the formation of a so-called caustic shadow in the spectrum of the MR waves.

By Rasmussen et al. (2006) the generation of zonal flow is illustrated in a simple fluid experiment performed in a rotating container with a radially symmetric bottom topography. It seems reasonable to simulate two-stream-like generation of zonal flows to confirm the theory of Rossby wave excitation provided in the present paper.

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