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ORIGINAL RESEARCH

# Electrostatic Solitary Waves

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**Abstract** Nonlinear ion acoustic waves using a fluid model in cylindrical coordinates for the propagation parallel to the magnetic field are investigated in a low-$\beta$ magnetized plasma. It is found that beside the existence of smooth regular solitons, cusp solitons also exist in plasmas when certain plasma conditions are satisfied. Characteristics of bipolar electric field solitary structures are also studied corresponding to the smooth regular solitons. Theoretical results agree with observations.

**Keywords** Electrostatic waves · Cusp solitons · Bipolar EFS structures

## Introduction

Ion acoustic and ion cyclotron waves in collisionless plasmas have been studied extensively both in unmagnetized and magnetized regimes using the ideal fluid equations [1–9]. The results showed that ion-acoustic and ion-cyclotron waves can evolve into nonlinear density periodic waves and density smooth solitons under different plasma conditions. Nonlinear electrostatic waves are responsible for the acceleration of up flowing ions and electron acceleration in the auroral region [10–12]. Numerous satellites have observed the bipolar Electric Field Solitary Structures (EFSs) parallel to the magnetic field throughout the space plasmas [13–16]. The structures were first observed in auroral zone by S3-3 satellite with velocities 50 km/s parallel to the magnetic field with the structure width of about 300 m [17]. POLAR observations of bipolar EFSs parallel to the magnetic field [13] showed the structure's amplitude $\sim$200 mV/m larger than any previous observations and revealed that the low altitude rarefactive ion-acoustic solitary waves propagating parallel to the magnetic field are associated with an increase in the width with increasing amplitude [18] which is in contrast to small amplitude 1D ion acoustic solitons theory which predicts the opposite relation [1, 5, 6] or for the fully nonlinear Pseudopotential technique where the Cartesian coordinates have been used [8, 9]. In all the theoretical studies cited above [1–9], the solitary solutions have been found in Cartesian geometry, in which propagation is either oblique or perpendicular to the background magnetic field. Maxon and Viecelli [19] studied small amplitude cylindrically symmetric ion-acoustic density solitary waves propagating radially inward.

In addition to these regular smooth solitons, solitons with cusp profiles have also been found in plasmas. Porkolab and Goldman [20], reported cusp solitons while studying the upper-hybrid waves perpendicular to the background magnetic field and two stream instabilities with warm two fluid equations. Shapiro [21] studied the modulational interaction of the lower-hybrid waves with a kinetic-Alfven mode and found the cusp solitons. Yinhua et al. [22] studied the nonlinear dust kinetic Alfven waves perpendicular to the external magnetic field by employing pseudopotential formulism and reported the smooth regular solitons as well as cusped solitons in the electron density

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corresponding to the singularities in the Sagdeev potential profiles in the subsonic regimes. Wei and Yinhua [23] also investigated the presence of cusp solitons along with the regular smooth solitons perpendicular to the external magnetic field with pseudopotential technique during their study of nonlinear lower hybrid waves in two-ion-species plasma. Observations of ion acoustic density fluctuations [24, 25] and density spiky pulses [26–28] have been reported by Freja satellite in the auroral upper ionospheric region.

In the present paper, we present ion fluid model in a low plasma $\beta$, which is the ratio of gas pressure to magnetic pressure ($8\pi p/B^2$), with cylindrical symmetry and derive the "Sagdeev or pseudopotential potential" for propagation exactly parallel to the external magnetic field. Linear analysis of the two fluid equations shows that the ion acoustic waves can evolve into nonlinear waves. It is found that not only smooth regular solitons but also cusp solitons exist in subsonic and supersonic regimes in plasmas, respectively. Characteristics of Bipolar EFSs structures and a comparison of theretical results with the observations are also presented. It is found that the theoretical results are in good agreement with the observational findings in auroral zone.

## Physical Formulation

The fluid consists of electrons and ions, in a low $\beta$ plasma. The magnetic field is directed along the $z$-axis. We are merely looking for electrostatic solutions and the magnetic field will be passively taken into account in the gyro-frequency. Using a cylindrical coordinate system and neglecting the electron inertia the fluid equations for the ions in cylindrical coordinates can be written as

$$\frac{\partial n}{\partial t}+\frac{\partial(nv_r)}{\partial r}+\frac{\partial(nv_z)}{\partial z}=-\frac{nv_r}{r} \tag{1}$$

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}=-\frac{1}{m_i n}\frac{\partial p}{\partial r}-\frac{e}{m_i}\frac{\partial \varphi}{\partial r}+\frac{v_\theta^2}{r}+\Omega_i v_\theta \tag{2}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}=-\frac{v_r v_\theta}{r}-\Omega_i v_r \tag{3}$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{m_i n}\frac{\partial p}{\partial z}-\frac{e}{m_i}\frac{\partial \varphi}{\partial z} \tag{4}$$

with

$$p = nT_i \tag{5}$$

$$n = n_i \approx n_e \approx n_0 \exp(e\varphi/T_e) \tag{6}$$

where $p$ is the thermal pressure, $e$ is the elementary charge, $\varphi$ is the electric potential, $v$ is the ion velocity and $n_0$ is constant number density. First, we linearize the set of (1–4), and obtain the linear dispersion relation as $\omega^2\left(\omega^2-aC_s^2k_z^2\right)\left(\omega^2-\Omega_i^2\right)=0$. Here $\omega$ is the wave frequency, the coefficient $a=T_i/T_e+1, C_s=(T_e/m_i)^{1/2}$is the ion acoustic velocity, $k_z$ is wave number, and $\Omega_i=eB_0/(m_i c)$is ion gyro frequency. From dispersion relation we obtain, $\omega^2=aC_s^2k_z^2$ and $\omega^2=\Omega_i^2$, which shows an ion acoustic wave can be excited in the plasma and consequently may develop into nonlinear waves.

## Nonlinear Analysis

We rewrite (1–4) in nonlinear form by introducing the stretched variable $\xi=(k_r r+k_z z-\omega t)\,.\,\Omega_i/\omega=(\alpha R+\gamma Z-M\tau)/M$ and dimensionless quantities: $N=n/n_0$, $\tau=\Omega_i t, R=r/\rho_i, Z=z/\rho_i, \boldsymbol{V}=\boldsymbol{v}/C_s$, $\Phi=e\varphi/T_e$, $M=v_p/C_s, v_p=\omega/k$, as

$$-\frac{dN}{d\xi}+\frac{\alpha}{M}\frac{d(NV_R)}{d\xi}+\frac{\gamma}{M}\frac{d(NV_Z)}{d\xi}=-\frac{NV_R}{R} \tag{7}$$

$$-\frac{dV_R}{d\xi}+\frac{\alpha V_R}{M}\frac{dV_R}{d\xi}+\frac{\gamma V_Z}{M}\frac{dV_R}{d\xi}=-\frac{a\alpha}{M}\frac{d(\ln N)}{d\xi}+\frac{V_\theta^2}{R}+V_\theta \tag{8}$$

$$-\frac{dV_\theta}{d\xi}+\frac{\alpha V_R}{M}\frac{dV_\theta}{d\xi}+\frac{\gamma V_Z}{M}\frac{dV_\theta}{d\xi}=-\frac{V_R V_\theta}{R}-V_R \tag{9}$$

$$-\frac{dV_Z}{d\xi}+\frac{\alpha V_R}{M}\frac{dV_Z}{d\xi}+\frac{\gamma V_Z}{M}\frac{dV_Z}{d\xi}=-\frac{a\gamma}{M}\frac{d(\ln N)}{d\xi} \tag{10}$$

where $N=\exp(\Phi)$, $\alpha=\sin\theta$, $\gamma=\cos\theta$ and $k_r$ and $k_z$ are the components of $K$ in the directions of $r$ and $z$, respectively. Now we consider the wave propagation along the magnetic field so that, $\alpha=0$ and $\gamma=1$. Integrating (10) with the boundary conditions when $\xi\to\infty$, $V=0$ and $N=1$, we can write

$$V_z=M\left[1\pm\sqrt{1-\frac{2a}{M^2}\ln N}\right] \tag{11}$$

By substituting (11) into (7–9), we can write (7–9), respectively, as

$$\frac{NV_R}{R}=\pm\left(-\sqrt{1-\frac{2a}{M^2}\ln N}+\frac{a}{M^2}\frac{1}{\sqrt{1-\frac{2a}{M^2}\ln N}}\right)\frac{dN}{dS} \tag{12}$$

$$\pm\sqrt{1-\frac{2a}{M^2}\ln N}\frac{dV_R}{dS}=\frac{V_\theta^2}{R}+V_\theta \tag{13}$$

$$\pm\sqrt{1-\frac{2a}{M^2}\ln N}\frac{dV_\theta}{dS}=-\frac{V_\theta V_R}{R}-V_R \tag{14}$$

We now solve the above set of equations using the same boundary conditions and obtain [29]

$$\frac{1}{2}\left(\frac{dN}{d\xi}\right)^2 + \Psi(N) = 0 \tag{15}$$

where

$$\Psi(N) = -\frac{\left[\left(N\sqrt{1-\frac{2a}{M^2}\ln N}-1\right)^2+\left(\frac{a}{M^2}-1\right)^2E_0^2\right]\left(N\sqrt{1-\frac{2a}{M^2}\ln N}\right)^2}{2\left[\frac{a}{M^2}-\left(1-\frac{2a}{M^2}\ln N\right)\right]^2} \tag{16}$$

where $E_0$ is the initial value of $E$. Equation (15) is the energy integral of a classical particle in a 1-D "potential well" and $\Psi(N)$ is so-called the "Sagdeev potential". The nonlinear solution from (16) can be obtained numerically when the parameters lead to a "Sagdeev potential" $\Psi(N) < 0$.

## Nonlinear Solution When $a/M^2 > 1$

By analyzing the properties of the "Sagdeev potential" $\Psi(N)$, we find that when the plasma parameters satisfy the condition $(a/M^2) > 1$ and $E_0 = 0$, $\Psi(N)$ has the properties:

$$\Psi(1) = 0, \quad \frac{d\Psi(1)}{dN} = 0, \quad \frac{d^2\Psi(1)}{dN^2} < 0, \quad \text{and} \quad \Psi(N_{\max}) = 0, \quad \frac{d\Psi(N_{\max})}{dN} > 0 \tag{17}$$

and $\Psi(N) < 0$ for $1 < N < N_{\max}$, where $N_{\max} = \exp\,[M^2/2a]$. In this case (16) has a solution corresponding to density hump soliton. These solitary waves are identified as subsonic ion-acoustic solitons as the wave speed $V_p$ is less than $C_s$ i.e., $M < 1$.

Figure 1 is plotted for different values of $a/M^2$ satisfying the condition (17). Here the ion to electron temperature ratio is taken as $10^{-1}$, and thus the parameter $a = 1.1$ throughout this paper for purposes of numerical work. It is to be noted that regular Sagdeev potential profile exists under the condition (17). The analogous particle in such a well will pass through the well and reach the opposite wall and get reflected there. The corresponding density hump soliton would then be a regular smooth soliton. Such smooth solitons are shown in Fig. 2, for the same parameters given in Fig. 1. From Fig. 2, we can see that both the width and amplitude of the regular soliton increase when the Mach number $M$ increases. In Fig. 3, maximum amplitude of the regular soliton is plotted as a function of Mach number $M$. We can note that the maximum amplitude of the regular solitons monotonically increases with $M$.

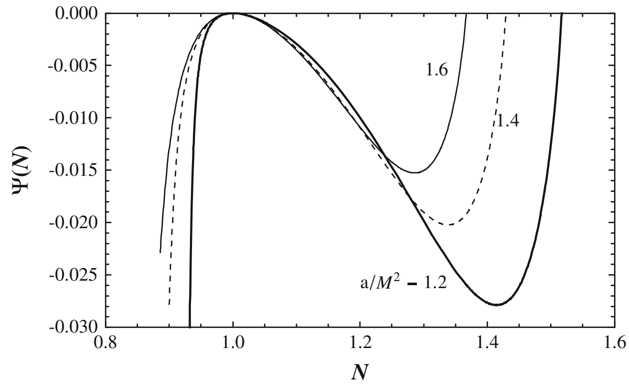


**Fig. 1** Profile of Sagdeev potentials $\Psi(N)$ for different values of $a/M^2$

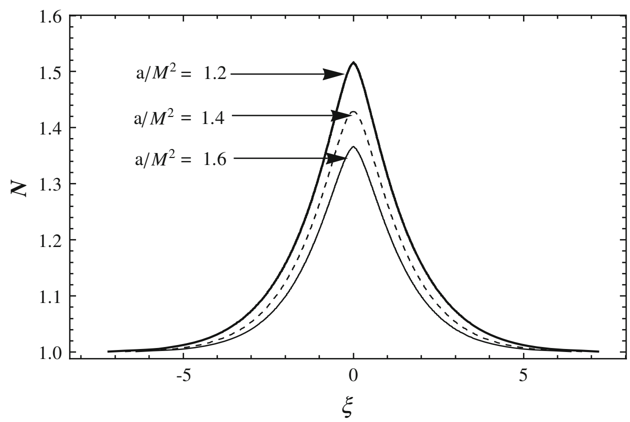


**Fig. 2** Regular smooth soliton structures corresponding to the Sagdeev potential $\Psi(N)$ in Fig. 1 for the same values of $a/M^2$

## Nonlinear Solution When $a/M^2 < 1$

Again by analyzing the properties of the "Sagdeev potential" $\Psi(N)$, we find that when the plasma parameters satisfy the condition $(a/M^2) < 1$ and $E_0 = 0$, $\Psi(N)$ has the properties:

$$\Psi(1) = 0, \quad \frac{d\Psi(1)}{dN} = 0, \quad \frac{d^2\Psi(1)}{dN^2} < 0, \quad \text{and} \quad \Psi(N_1) = -\infty \tag{18}$$

Figure 4 is plotted for different values of Mach number $M$ satisfying condition (18). We note that there exists a Sagdeev potential of infinite depth when $\Psi(N_1) \rightarrow -\infty$ at $N_1$ where $1 < N_1 < N_{\max}$. Ideally, a particle would start slowly from $N = 1$, attains infinite speed at $N = N_1$ and will pass the infinite well with infinite speed and then slow down until it reaches the opposite wall. The particle then returns to the point $N = 1$ in a symmetrical way. The corresponding solitary wave would then be a cusp like

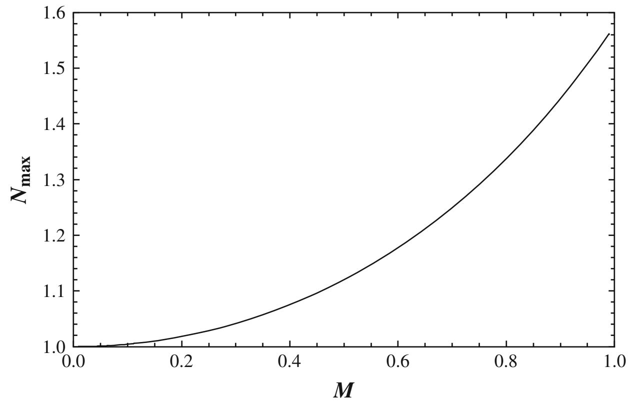


**Fig. 3** Variation of maximum amplitude of the soliton with the Mach number $M$ for regular solitons

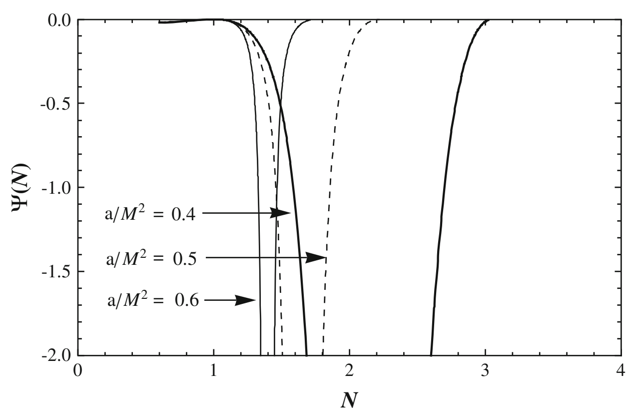


**Fig. 4** Profile of Sagdeev potentials $\Psi(N)$ for different values of $a/M^2$

solitons but with a smooth top [30]. At the two points around $N = N_1$ on its shoulders the Sagdeev potential has infinite derivatives but all the physical quantities remain continuous. On the other hand, in the infinite well the particle motion is extremely sensitive to any deviation from the ideal condition. If we assume that it is reflected at point ($N_1$), we will get corresponding cusped density hump solitons [20, 23]. This type of soliton is quite different from the regular smooth solitons corresponding to the Sagdeev potential in Fig. 2. Figure 5 depicts the cusped density hump solitons corresponding to the Sagdeev potential in Fig. 4 for the same parameters. We can see that both the amplitude and width of the cusped solitons increase when Mach number $M$ increases. These cusp solitons are identified as supersonic ion-acoustic solitons as the wave speed $V_p$ is larger than $C_s$ i.e., $M > 1$. In Fig. 6, max amplitude of the cusp soliton corresponding to the point $N = N_1$ of the Sagdeev potential is plotted as a function of Mach number $M$. We can note that the amplitude of the cusp solitons increases almost linearly with $M$.

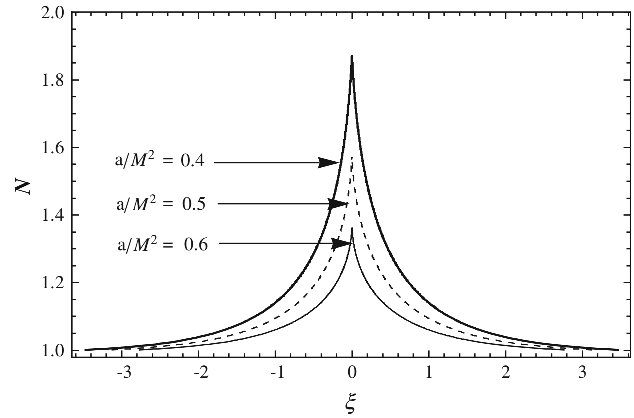


**Fig. 5** Cusp soliton structures corresponding to the Sagdeev potential $\Psi(N)$ in Fig. 4 for the same values of $a/M^2$

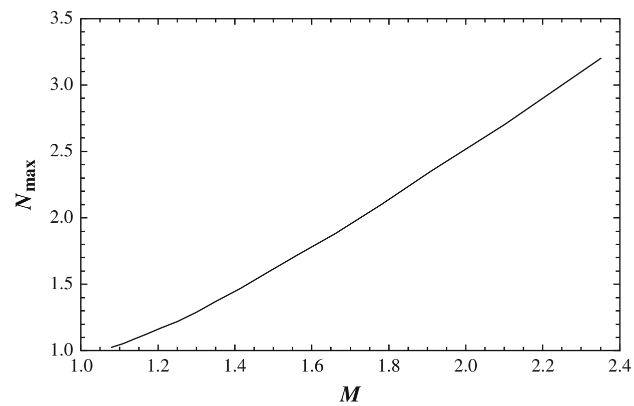


**Fig. 6** Variation of maximum amplitude of the cusp soliton corresponding to $N_1$ with the Mach number $M$

## Bipolar Electric Field Solitary Structures

By definition we can write electric field as $E = -\partial_\xi \Phi = -(1/N)(\partial_\xi N)$. Therefore, corresponding to the regular smooth solitons, we can obtain the bipolar EFSs as shown in Fig. 7. We can see that the amplitude of bipolar EFSs increases with the Mach number (see Fig. 8) which is in agreement with the auroral zone observations [13]. We can also obtain the electric field profiles corresponding to the cusp solitons, which will be consisting of two adjacent spikes, one positive and one negative. Such an electric field wave form is shown in a schematic way in Fig. 9.

## Analysis and Conclusion

Solitons with cusp profiles occur because of a balance of the dispersive and nonlinear effects close to the wave breaking point, and have been found in both fluids and plasmas [20–22, 30]. In our theoretical model, we have

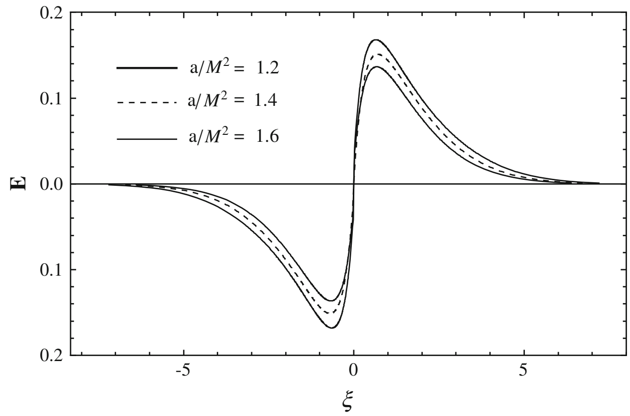


**Fig. 7** Electric field structures corresponding to regular smooth solitons

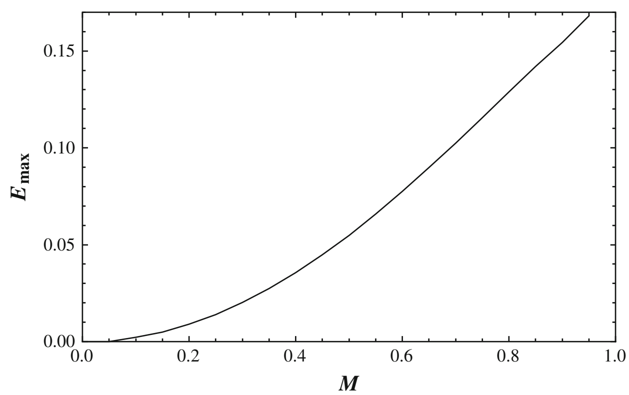


**Fig. 8** Variation of maximum amplitude of the electric field structures with the Mach number *M*

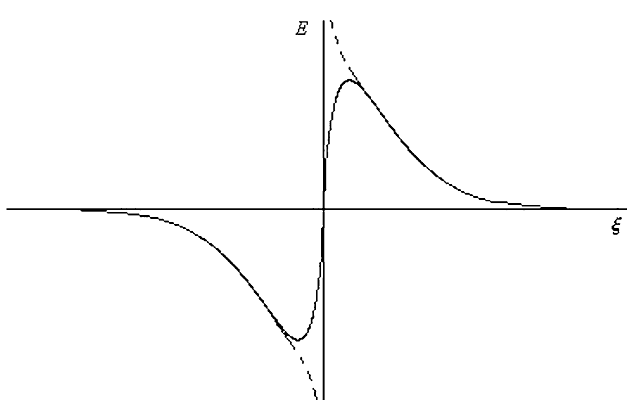


**Fig. 9** Electric field structure corresponding to the cusp solitons shown in *dashed line*

assumed that the phase velocity lies between ion and electron thermal velocities so that Landau damping can be neglected. However, in real situations Landau damping always exists which prevents the Sagdeev potential to become infinitively deep. Moreover, the cusps where the gradients are very large are expected to be smoothed out somewhat by higher order nonlinearities or dissipative processes such as those arising from shear or turbulent viscosity [22, 23], should limit the amplitude of spiky electric field profiles corresponding to the cusp solitons.

Considering the observations of POLAR and FAST satellite in auroral zone, the ion cyclotron frequency $f_{ci}$ is about 100 Hz and the electron temperature ranges from 6 to 25 eV. Therefore, the acoustic speed $C_s$ will be from 30 to 63 km/s. From the theoretical model, we can write the width of the structure, time duration of the structure and the amplitude of electric field as $\xi = (z/v_p - t)$, $t = (1/M - \xi)$ and $E = (T_e/e).(\Omega_i/(C_s M).|E_M|$, respectively. Here, $|E_M|$ is the normalized amplitude of the bipolar EFSs from our model and vp is the phase velocity which ranges from 3 to 60 km/s according to our model. If we consider the possible values of $\xi$ 10–15, $M$ from 0.1 to 0.95 and $|E_M|$ from 0.01 to 0.17, the width of the bipolar structures would be from 10 m to 1.4 km, the time duration lie from 3.33 to 23 ms and the amplitude would be 12–238 mV/m. According to the observations, the width of the bipolar EFSs is about 300 m, the duration lies between 2 and 20 ms and the amplitude is between 50 and 200 mV/m [13, 14, 18, 31]. Therefore, our results are in good agreement with the observations.

The results from our model show that the cusp solitons in the supersonic whereas smooth regular solitons and bipolar EFSs structures exist in the subsonic regimes for propagation parallel to the magnetic field. Both amplitude and width of these solitary and electric field profiles increase with the Mach number *M*. This is consistent with the observations [13]. In our model, the amplitude of solitary structures increases with the width of the soliton, which is also consistent with the observations [18]. Comparison of Figs. 2 and 5 shows that the amplitudes of supersonic cusp solitons are larger but their widths are smaller than that of subsonic regular solitons. Events of spiky density pulses have been reported by Freja spacecraft in the auroral region of Earth's ionosphere [24–28] which may be explained by modeling them with the cusp solitons obtained in this work. Therefore, our results have a direct bearing in understanding the properties of ion acoustic regular and cusp solitons, and bipolar EFSs in space plasmas.

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