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Formation of acoustic nonlinear structures in non-Maxwellian trapping plasmas

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Abstract

In this paper, expressions of number densities for electron trapping for generalized (r, q), kappa, and Cairns distribution functions, respectively, are reported using the approach adopted by Landau and Lifshitz for Maxwellian trapping of electrons. For illustrative purposes, dispersive and dissipative equations for ion-acoustic waves (IAWs) are obtained in the presence of non-Maxwellian trapped electrons in the small amplitude limit. The solutions of the modified dispersive and dissipative nonlinear equations are reported and graphical analysis is given to present a detailed comparison of non-Maxwellian and Maxwellian trapping. The results presented here are, to the best of authors' knowledge, are a first attempt of this kind. It is expected that the present investigation will unravel new horizons for future research and encourage the researchers to search for the nonlinear structures presented in this paper in the satellite data.

Keywords: (Non)-Maxwellian trapped electrons, Trapping plasmas; Superthermal electrons; Nonthermal electrons; Nonextensive electrons; Generalized (r, q) electrons; Trapped KdVB with power law; Solitons and shocks.

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I. INTRODUCTION

Electrostatic acoustic perturbations are the most thoroughly investigated plasma modes [1]-[7]. It is one of the three fundamental modes of a plasma and its characteristic signature is that it compresses the plasma. Since it is an electrostatic mode, therefore, magnetic perturbation is absent here. The nonlinear structure of the ion-acoustic waves (IAWs) forms for the dispersive IAWs where the dispersion comes from the charge separation effects. Solitons are amongst the most important nonlinear structures that propagate in a plasma [8–11]. They are formed due to the balance of the dispersion and nonlinearity. The particles can get trapped in these structures which result in the readjustment of energy and momentum in the system. When the dissipation is presented, shocks can be formed in the system and in the presence of both dispersion and dissipation, mixed partial nonlinear differential equations like one-dimensional perturbation Korteweg-de Vries Burgers (KdVB) equation, two-dimensional perturbation Kadomtsev Petviashvili Burgers (KPB) equation, three-dimensional perturbation Zakharov Kuznetsov Burgers (ZKB) in unmagnetized and magnetized plasmas are obtained which also admit shock solutions [12]-[21].

Trapping of particles is another field where the nonlinear effects manifest themselves. In trapping, the particles are captured in a peculiar region of phase space. This results in the particles bouncing back and forth and forming closed trajectories. Bernstein et al. [22] put forth the idea of trapping and observed that the nonlinear dynamics of the plasma get significantly altered with the inclusion of particle trapping effects. Gurevich [23] used a different approach, termed as adiabatic trapping, and found that the trapping alters the power of nonlinearity. Extensive study of the propagation of solitary and shock structures in the presence of trapped charged particles in plasmas has been carried out [24]-[32]. Shah et al. [26] studied the effect of adiabatic electron trapping on the obliquely propagating Alfvén waves and theoretically obtained regimes of existence of the solitons. It was found that the obliqueness angle θ and plasma β have a significant impact on the soliton formation in the system. Masood et al. [31] studied solitons and shocks in weakly relativistic electron-ion $(e-i)$ plasmas with electron trapping and obtained modified nonlinear evolution equations. The authors used an extension of the tanh method to get the solutions of these modified nonlinear partial differential equations (NLPDEs). Most importantly, it was found that length scales of the formation of nonlinear structures were different for different types of nonlinearities present in the system. Demiray et al. [33] studied electrostatic waves on the electron time scale, i.e., EAWs with the hot electrons following the vortex distribution in cylindrical and spherical geometries and reported new analytical solution and made its comparison with the numerical solution. The authors reported a good agreement between analytical and numerical results. Recently, Naira et al. [32] studied the radially ingoing and outgoing solitons and shocks in a weakly relativistic plasma having trapped electrons. The authors found that the solitons propagating in a nonplanar geometry exhibit a phase shift both for the ingoing and outgoing cases unlike their dissipative counterparts. The difference in nonlinear velocities was found to be the reason behind it.

The spacecraft observations from different regions of space plasmas indicated the existence of charged particles whose behavior deviates significantly from the Maxwellian behavior. These include kappa, Cairns, and double spectral index distributions, respectively. Vasyliunas [34] used kappa distribution to get a good agreement with the OGO 1 and OGO 3 data. Ever since, kappa distribution has been shown to agree well with the observations in space plasmas [35]-[40] (and the references therein). Viking and Freja satellites [41, 42] reported the rarefactive ion-acoustic (IA) solitary structures which could not be explained theoretically in the presence of Maxwellian electrons. Cairns et al. [43], therefore, proposed a new distribution function for electrons to obtain dips in solitary structures in accordance with Freja and Viking observations [41, 42].

Many satellite observations have alluded to the fact that the electrons are inertialess distributed in space plasmas with modified shapes both for low and high energy electrons. Qureshi et al. [44] devised a new distribution function called the generalized (r, q)-distribution (to reduce the similarity, we can say the (r, q)-distribution) function consisting of two-parameters to account for the modified behavior of low- and high-energy of the electrons to get a good agreement with the observations of the electron distribution functions in space plasmas [45, 46]. The (r, q)-distribution can be recovered both the Maxwellian and superthermal/kappa distributions in the limiting cases [47]. The parameter r controls the low energy electron behavior whereas the parameter q alters the tail population. It has satisfactorily explained many observations in space plasmas which other distributions could not [45]-[49]. Also, the (r, q)-distribution has been applied to explicate the data in the Earth's magnetosphere [47, 49]. The effect of the $r, q)$-distribution on a variety of plasma waves has also been studied in the recent past [50]-[62]. Interestingly, it was shown by Shah et al [50] that like the Cairns distribution, the (r, q)-distribution can also give rarefactive ion acoustic solitons and, therefore, present an alternate framework to study the rarefactive IA solitons.

In this paper, we present the expressions of number density of electrons with trapping using kappa, Cairns and the (r, q)-distribution and also show how these expressions reduce to Maxwellian expression in the limiting case. For illustrative purposes, we have studied non-linear IA solitary and shock waves in the presence of kappa, Cairns and (r, q)-distributed trapped electrons. We have made a detailed comparison of the difference for all the afore-mentioned distributions for trapped electrons.

The backbones of this paper is introduced in the following form: In Sec. 2, the mathematical formulation of finding the number densities for trapped electrons for different distributions is presented. In Sec. 3, the nonlinear evolution equations with the inclusion of trapped electrons are obtained. In Sec. 4, we obtain the solutions of trapped equations by using the modified tangent hyperbolic method. In Sec. 5, the effects of the adiabatic electron trapping in non-Maxwellian plasmas are discussed in detail. Finally, Sec. 6 is kept for summary and conclusion.

II. MATHEMATICAL FORMULATION

We present here the theoretical framework of finding the expression of number density of the trapped electrons in the presence of the (r, q)-distribution function. Our procedure closely follows the one adopted by Landau and Lifshitz [63] where they find an expression for the adiabatic electron capture in the presence of Maxwellian distribution. By assuming that the characteristic time of the variation of the electric field is much smaller than the electron mean free time, it is justified to consider the field to be stationary during the passage of electron through it. In the same vein, the electron distribution which, in our case, happens to be the (r, q)-distribution is also assumed to be stationary. We consider here one-dimensional adiabatic trapping and the field potential ϕ is assumed to depend only on the coordinate x. This assumption makes the motion in the y and z directions unimportant and, therefore, we consider the distribution function to depend only on the momentum p_x [63] which reads as follows

$$f_{r,q} = A[1 + \frac{1}{q-1}(\frac{p_x^2}{mb^2T_e} - \frac{2\,|U|}{b^2T_e})^{r+1}]^{-q}, \tag{1}$$

with

$$A = \frac{\Gamma[q]}{2\sqrt{b^2 m T_e}\left(\frac{1}{-1+q}\right)^{-\mathbb{R}}\Gamma[q-\mathbb{R}]\Gamma[1+\mathbb{R}]}, \tag{2}$$

$$b = \frac{\sqrt{3}(q-1)^{-\mathbb{R}}\sqrt{\Gamma[q-\mathbb{R}]}\sqrt{\Gamma[1+\mathbb{R}]}}{\sqrt{\Gamma[q-3\mathbb{R}]}\sqrt{\Gamma[1+3\mathbb{R}]}}, \tag{3}$$

where $\mathbb{R} = \frac{1}{2(1+r)}$.

In the light of above assumptions, we can find the spatial distribution of electrons by adding the number of electrons for the trapped and untrapped electrons in the following manner

$$n_e = 2A\left(\int_0^{p1} f(0)dp_x + \int_{p1}^{\infty}[1+\frac{1}{q-1}(\frac{p_x^2}{mb^2T_e} - \frac{2\,|U|}{b^2T_e})^{r+1}]^{-q}dp_x\right), \tag{4}$$

where $p1 = \sqrt{2m\,|U|}$ and $U = -e\phi$.

Expansion of the potential and evaluation of cumbersome integrals denotes the following expression of total number density of electrons for the trapped (r,q)-distribution function

$$n_e = 1 + \Theta_{r,q}\phi + \Lambda_{r,q}\phi^{r+\frac{3}{2}} + \gamma_{r,q}\phi^2, \tag{5}$$

with

$$\Theta_{r,q} = \frac{\left(\frac{1}{-1+q}\right)^{2\mathbb{R}}\Gamma[1-\mathbb{R}]\Gamma[q+\mathbb{R}]}{b^2\Gamma[q-\mathbb{R}]\Gamma[1+\mathbb{R}]}\ , \tag{6}$$

$$\Lambda_{r,q} = \frac{2^{\frac{-1}{2}+r}q\,(-1+q)^{-\mathbb{R}(3+2r)}\,(4+r+3r^2+4r^3+4r^4)\Gamma[q]}{b^{3+2r}(1+r)(-1+2r)(1+2r)\Gamma[q-\mathbb{R}]\Gamma[2+\mathbb{R}]}, \tag{7}$$

$$\gamma_{r,q} = \frac{3(-1+q)^{-2}\Gamma[-3\mathbb{R}]\Gamma[q+3\mathbb{R}]}{2b^4\Gamma[\mathbb{R}]\Gamma[q-\mathbb{R}]}, \tag{8}$$

where $\phi = |U|\,/T_e$.

A. Limiting cases of the generalized (r,q) distribution

The expression of the trapped electron number density for (r,q)-distribution function (5) reduces to the trapped electron kappa distribution if $r = 0$ and $q \longrightarrow \kappa$ as follows

$$n_e = 1 + \Theta_\kappa\phi + \Lambda_\kappa\phi^{\frac{3}{2}} + \gamma_\kappa\phi^2, \tag{9}$$

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with

$$\Theta_\kappa = \frac{\left(\kappa - \frac{1}{2}\right)}{\left(\kappa - \frac{3}{2}\right)},\ \gamma_\kappa = \frac{\left(\kappa^2 - \frac{1}{4}\right)}{2\left(\kappa - \frac{3}{2}\right)^2},$$
$$\Lambda_\kappa = \frac{-4\Gamma[\kappa]}{\left(3\sqrt{\pi}\kappa^{1/2}\left(-\frac{3}{2}\kappa + 1\right)^{\frac{3}{2}}\Gamma\left[\kappa - \frac{1}{2}\right]\right)}.$$

Also, for $r = 0$ and $q \longrightarrow \infty$, we obtain the following normalized electron number density for the trapped Maxwellian distribution function [63]

$$n_e = 1 + \Theta_m\phi + \Lambda_m\phi^{\frac{3}{2}} + \gamma_m\phi^2, \tag{10}$$

where $\Theta_m = 2\gamma_m = 1$ and $\Lambda_m = -4/3\sqrt{\pi}$.

B. Cairns distribution

The Cairns distribution function is given by [43]

$$f = \frac{N_0}{(1+3\alpha)\sqrt{2\pi m T_e}}[1 + 4\alpha(\frac{p_x^2}{2mT_e} - \frac{|U|}{T_e})^2]e^{-\left(\frac{p_x^2}{2mT_e} - \frac{|U|}{T_e}\right)}, \tag{11}$$

and by employing the same formalism that are used above for the (r, q)-distribution function, we get the following expression for the total electron number density for trapped Cairns distribution function

$$n_e = 1 + \Theta_c\phi + \Lambda_c\phi^{\frac{3}{2}} + \gamma_c\phi^2, \tag{12}$$

with

$$\Theta_c = 1 - \beta,\ \Lambda_c = -\frac{4}{3}(1+3\alpha)\sqrt{\pi},$$
$$\gamma_c = \frac{1}{2},\ \beta = \frac{4\alpha}{(1+3\alpha)},$$

where α is the percentage of the nonthermal electrons. Note that the trapped Cairns distribution (12) has two shoulders and cannot be obtained from the (r, q)-distribution in any limit and, therefore, it is obtained by the method as outlined by Landau and Lifshitz [63].

III. NONLINEAR IAWS WITH NON-MAXWELLIAN TRAPPED ELECTRONS

To illustrate the significance of the trapping in a non-Maxwellian plasma, we consider one-dimensional unmagnetized viscous $e-i$ plasma and study the nonlinear acoustic perturbations in a plasma with warm ions in the presence of trapped electron under the assumption:

$\omega/k << \upsilon_{Ti} << \upsilon_{Te}$, where υ_{Ti} and υ_{Te} represent the ions and electrons thermal velocities, respectively, and ω/k denotes the phase velocity of the IAWs. Given below are the model equations:

$$\partial_t n_i + \partial_x(n_i u_i) = 0, \tag{13}$$

$$\partial_t u_i + u_i \partial_x u_i = -\partial_x \phi - \frac{\sigma}{n_i}\partial_x p_i + (2\eta + \eta_{II})\partial_x^2 u_i. \tag{14}$$

The quantities n_i, u_i, and ϕ are normalized by n_{e0}, $C_s = \sqrt{T_e/m_i}$ and T_e/e, respectively, $\eta = \kappa/(\lambda_{De}C_s)$ is the ion normalized kinematic viscosity, $\eta_{II} = \mu/(\lambda_{De}C_s)$ is the ion normalized volume viscosity coefficient also called the second coefficient of the viscosity, and $\sigma = T_i/T_e$ gives the ion temperature ratio. The evolution equation of the ion pressure is given by

$$\partial_t p_i + u_i \partial_x p_i + 3p_i \partial_x u_i = 0. \tag{15}$$

The ion pressure p_i is scaled by $n_i T_i$ while the spatio-temporal coordinates (x, t) are normalized by $\lambda_{De} = \sqrt{T_e/4\pi n_{e0}}$ and λ_{De}/C_s, respectively.

The total normalized electron number density for distribution functions under discussion can be written as

$$n_e = 1 + \Theta_d \phi + \Lambda_d \phi^{r+3/2}, \tag{16}$$

where the subscript d represents the different distributions such that (r, q), kappa (κ), Maxwellian (m), and Cairns (c). For the problem at hand, we will limit ourselves to the $3/2$ power of the perturbed potential in the expression of electron number density as we are looking for trapped KdV Burgers (TKdVB), trapped KdV (TKdV), and trapped Burgers (TB) equations for the nonlinear IAWs. The system is closed by the following Poisson's equation

$$\partial_x^2 \phi = n_e - n_i. \tag{17}$$

To study one-dimensional nonlinear excitations in an unmagnetized viscous $e-i$ plasma with non-Maxwellian trapped electrons, the following stretching for the space and time coordinates in terms of spectral index r is employed [4]

$$\begin{cases} \zeta = \epsilon^{(2r+1)/4}(x - \lambda t), \ \tau = \epsilon^{3(2r+1)/4} t, \\ \eta = \epsilon^{(2r+1)/4}\eta_0, \ \eta_{II} = \epsilon^{(2r+1)/4}\eta_{0II}, \end{cases} \tag{18}$$

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where $\epsilon << 1$ indicates the expansion parameter and the phase velocity λ is normalized by C_s. The physical quantities (n_i, u_i, p_i, ϕ) are expanded as

$$\begin{cases} n_i = 1 + \epsilon n_{i1} + \epsilon^{r+3/2} n_{i2} + \cdots, \\ u_i = \epsilon u_{i1} + \epsilon^{r+3/2} u_{i2} + \cdots, \\ p_i = 1 + \epsilon p_{i1} + \epsilon^{r+3/2} p_{i2} + \cdots, \\ \phi = \epsilon \phi_1 + \epsilon_2^{r+3/2} \phi_2 + \cdots. \end{cases} \tag{19}$$

Substituting Eqs. (18)-(19) into Eqs. (13)-(17) and collecting the different powers of ϵ, we obtain some reduced equation which the lowest-order in ϵ gives us

$$\begin{cases} n_{i1} = \Theta_d \phi_1, \\ u_{i1} = \Theta_d \lambda \phi_1, \\ p_{i1} = 3\Theta_d \phi_1, \end{cases} \tag{20}$$

and

$$\lambda = \sqrt{3\sigma + \frac{1}{\Theta_d}}. \tag{21}$$

which describes the linear propagation of ion sound excitations.

For the next higher-order in ϵ, we get

$$\begin{cases} \partial_\zeta u_{i2} - \lambda \partial_\zeta n_{i2} + \partial_\tau n_{i1} = 0, \\ -\lambda \partial_\zeta u_{i2} + \partial_\zeta \phi_2 + \sigma \partial_\zeta p_2 - (2\eta_0 + \eta_{0II}) \partial_\zeta^2 u_{i1} + \partial_\tau u_{i1} = 0, \\ 3\partial_\zeta u_{i2} - \lambda \partial_\zeta p_2 + \partial_\tau p_1 = 0, \\ \partial_\zeta^2 \phi_1 - \Lambda_d \phi_1^{r+3/2} - \Theta_d \phi_2 + n_{i2} = 0. \end{cases} \tag{22}$$

Now, by removing the second-order variables $(n_{i2}, u_{i2}, p_{i2}, \phi_2)$ in system (22) and using relations given in system (20), we finally get the following TKdVB equation for the IAWs with non-Maxwellian trapped electrons

$$\partial_\tau \Phi + A\Phi^{r+1/2} \partial_\zeta \Phi - B\partial_\zeta^2 \Phi + C\partial_\zeta^3 \Phi = 0, \tag{23}$$

with

$$\begin{cases} A = -\frac{(r+3/2)\lambda\Lambda_d}{2(1+3\sigma\Theta_d)}, \\ B = \frac{(2\eta_0+\eta_{0II})\lambda^2}{2(1+3\sigma\Theta_d)}, \\ C = \frac{\lambda}{2(1+3\sigma\Theta_d)}. \end{cases} \tag{24}$$

where $\Phi \equiv \phi_1$ and the coefficients of the nonlinearity, dissipative, and dispersive terms are, respectively, denoted by A, B, and C.

In order to find a general analytical solution to the TKdVB Eq. (23), the extended version of tangent hyperbolic (tanh) method is applied by introducing the transformation $\xi = k\left(\zeta - V\tau\right)$ [64]. Accordingly, the following shock solution to the TKdVB Eq. (23) is obtained

$$\Phi(\zeta,\tau) = \left(\frac{-(4r+10)(2r+3)B^2}{4(2r+9)^2 AC}\right)^{\frac{2}{2r+1}} \left[1 - \tanh\{k\left(\zeta - V\tau\right)\}\right]^{\frac{4}{2r+1}}, \tag{25}$$

with

$$\begin{cases} k = -(2r+1)B/2(2r+9)C, \\ V = -4(2r+5)B^2/(2r+9)^2 C, \end{cases}$$

where (V,k) are, respectively, the velocity and wave number of the traveling wave. Remember that the solution (25) satisfies Eq. (23).

For dispersionless IAWs, the coefficient of the dispersion term C vanishes and the TKdVB Eq. (23) reduces to and the following dissipative nonlinear equation, i.e., TB equation for the IAWs with non-Maxwellian trapped electrons

$$\partial_\tau \Phi + A\Phi^{r+1/2}\partial_\zeta \Phi - B\partial_\zeta^2 \Phi = 0. \tag{26}$$

Following the extended version of the tanh method mentioned above [64], the following shock solution of Eq. (26) is obtained

$$\Phi(\zeta,\tau) = \left[\frac{(2r+3)Bk}{A(2r+1)}\right]^{\frac{2}{2r+1}} \left(1 - \tanh\left[k\left(\zeta - V\tau\right)\right]\right)^{\frac{2}{2r+1}} \tag{27}$$

where $V = 4Bk/(2r+1)$. This solution (solution burgers) satisfies Eq. (26).

For $B = 0$, the TKdVB Eq. (23) reduces to the following dispersive nonlinear equation, i.e., TKdV equation for the IAWs with non-Maxwellian trapped electrons

$$\partial_\tau \Phi + A\Phi^{r+1/2}\partial_\zeta \Phi + C\partial_\zeta^3 \Phi = 0. \tag{28}$$

The solitary wave solution to Eq. (28) reads

$$\Phi(\zeta,\tau) = \left(\frac{(4r+10)(2r+3)Ck^2}{A(2r+1)^2}\right)^{\frac{2}{2r+1}} \left[\sec h\{k(\zeta - V\tau)\}\right]^{\frac{4}{2r+1}}, \tag{29}$$

where $V = 16Ck^2/(2r+1)^2$. This solution (solution KdV) satisfies Eq. (28). It must be mentioned here that many papers have recently been published that present interesting new solutions of a wide variety of the nonlinear partial differential equations [65]-[70].

IV. RESULTS AND DISCUSSION

Now, we shall numerically explore the effect of the adiabatic electron trapping in a non-Maxwellian plasma (including double spectral index (r, q), kappa, and Cairns) on the profile of both solitary and shock wave solutions. Note that the expression of the trapped electron density for the generalized (r, q) distribution can be recovered the expressions of number density for trapped kappa and Maxwellian distributions in the limiting cases, however, the expression for the number density of the trapped Cairns distribution is obtained separately. As we know, the spectral indices r and q alter the population of low and high energy electrons, respectively.

Regarding the physical application of this work, we would like to mention that kappa and Cairns distributions have satisfactorily explained many observed phenomena in space plasmas as mentioned in the introductory section. It was shown by Schippers et al. [71] that the magnetosphere of Saturn is replete with superthermal/kappa electrons. Henning et al. [72] showed that the nonthermal features of the two electrons population observed in the Saturnian magnetosphere affects on the behavior of the nonlinear wave propagation. The authors [72] also showed that the two electrons population found in Saturn are best modelled by dual kappa distribution. It has already been mentioned that Cairns distribution satisfactorily explains the dip acoustic structures observed by Viking and Freja satellites [41]-[43]. The (r, q)-distribution has also been successfully applied to explain the phenomena in the terrestrial magnetosheath near the magnetic cusp regions because they fit the flat-topped electron distributions observed in these regions. For instance, Cluster satellite has observed such flat-top distributions in the Earth's magnetosheath [45]-[49]. ARTEMIS spacecraft has made similar observations in the magnetotail [73]. Observations of these distribution functions in the magnetic reconnection region [74] and bow shock of Venus [75] have also been reported. The parameters are chosen in such a way that they are good for presenting a general application framework. As regards the (r, q)-distribution, we have taken guidance from previous studies to select the values of r and q which agreed well with the observed data for the electron distribution and also with the view to explore the whole range which would be beneficial for a general study like this. Moreover, the spatial scale ξ is normalized by the Debye length and since it contains the temperature and number density, therefore, the spatial scale over which the structures shown in this paper would form would vary for

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different regions of space plasmas.

A. Analysis of nonlinear structures of the TKdVB Eq. (23)

Here, we explore parametric variation on the profile of the nonlinear wave solutions to the TKdVB Eq. (23). Figure 1a explores the effect of increasing r or increasing flatness of the electron distribution function on the solution admitted by the TKdVB Eq. (23). It is observed that the increase of flatness of the electron distribution function leads to the enhancement of the shock strength. Figure 1b exhibits the behavior of the shock structure admitted by the TKdVB Eq. (23) by altering the electron population in the tail of the distribution function, i.e., varying q. It is found that the decrease of q leads to the reduction of the shock strength. This implies that enhancing the tail population or enhancing the percentage of energetic electrons in low phase space density regions enervates the ion dynamics driven shock waves.

Figure 2 makes a comparison of the non-Maxwellian and Maxwellian distributions. Note that the values of kappa in space plasmas are generally taken to be $2-6$ and that have been taken into consideration while plotting this and the other graphs that include kappa contribution in the paper. The higher values of kappa make the distribution function get closer to the Maxwellian distribution function. It can be seen that the shock strength is optimum for flat-topped, intermediate for Cairns whereas it is minimum for the kappa distribution of the trapped electrons. One can see that the increase of r and the nonthermal parameterα lead to the increase of the percentage of low energy electrons and the enhancement of the shock strength. The kappa distribution, which modifies the tail or the percentage of high energy electrons, is found to mitigate the shock strength just like the parameter q of the (r, q)-distribution function (see Fig. 1b). This fortifies the view that changing the percentage of low energy electrons have a much more pronounced effect on the shock strength by comparison with the change produced by altering the percentage of the electrons in the region of low phase space density or the high energy electrons. There is another possibility of taking negative value of r that mimics a spiky distribution function. If we take $r < 0$, we find that the solution of the TKdVB Eq. (23) does not allow any shock formation. It is worth mentioning that the solution of the TKdVB Eq. (23) allows the shock formation only for $r > 1/2$.

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B. Analysis of nonlinear structures of the TB Eq. (26)

Figure 3a investigates the effect of increasing percentage of low energy electrons on the propagation characteristics of shock structure admitted by the TB Eq. (26). Importantly, it is observed that the range of r for which shock formation is possible for Burgers equation is $-1/2 \leq r \leq 1/2$, which is very different from the TKdVB Eq. (23). Not only the range for flat-topped (i.e. positive values of r) is different but here the formation of shocks for spiky distribution (i.e. negative values of r) is also possible. It is observed that enhancing the flatness of distribution function in the region of high phase space density (i.e. increasing r) augments the shock strength similar to what was observed in the case of the TKdVB Eq. (23), however, the shock strength of the TB Eq. (26) is found to be smaller in magnitude by comparison with the TKdVB shock. This is anti-intuitive as one expects the Burgers shock to be larger in magnitude by comparison with its TKdVB counterpart due to absence of dispersion. A closer look will tell that this is due to the different ranges of r for which shocks are obtained for the TKdVB and TB Eqs. (23) and (26). The reason of higher shock strength of the TKdVB Eq. (23) is that we use larger values of r for the TKdVB Eq. (23) than TB Eq. (26). The shock strength, however, mitigates with the increasing percentage of high energy electrons in the tail (i.e. decreasing q) as can be seen from Fig. 3b. Other than that, the discussion of Fig. 3a holds here as well regarding the values of r for which the TB Eq. (26) admits shock solutions.

Figure 4a presents a comparison of the different distributions for the TB Eq. (26). Unlike the TKdVB Eq. (23), it can be seen that the shock strength for the TB Eq. (26) is maximum for Cairns whereas it is minimum for kappa distributed trapped electrons. Barring the flat-topped curve, all the figures show a stronger shock by comparison with their TKdVB counterparts due to the reasons outlined above. Since the TB Eq. (26) allows the shock formation for negative values of r (i.e. for the spiky distribution), Fig. 4b shows the comparison of spiky distribution with kappa, Cairns and Maxwellian distributions for electron trapping. Here we find that the shock strength is still maximum for the Cairns distribution, however, the minimum shock strength in this case is for spiky distribution.

C. Analysis of nonlinear structures of the TKdV Eq. (28)

As regards the TKdV Eq. (28), it is found to allow the solitary wave formation for the same range of r as that of the TB Eq. (26). The scarcity of space does not allow us to show the behavior of the solitary wave with increasing r and q indices, however, it is adequate to say that they show similar trend to that of the TKdVB Eq. (23) and the TB Eq. (26), i.e. the increasing r and q values enhance the amplitude of the nonlinear ion sound excitations. Figures 5a and 5b draw a comparison of non-Maxwellian and Maxwellian distribution for the IA solitary wave. It is observed that solitary wave has maximum amplitude for Cairns distribution and minimum for kappa distribution when the value of r is chosen to be positive as can be seen in Fig. 5a. However, when we choose $r < 0$ that corresponds to the spiky distribution, the amplitude of the solitary wave is found to be minimum for spiky distribution instead of kappa distribution (see Fig. 5b).

Figures 6a and 6b show the effect of increasing η (i.e., kinematic viscosity) on the strength of shock of the TKdVB Eq. (23) for (r, q) and kappa distributions, respectively. It is noticed that increasing η augments the shock strength. The trend is similar for the TB Eq. (26) for both generalized (r, q) and kappa distribution as shown in Figs. 7a and 7b. However, due to different ranges of r it can be seen that, for double index distribution, the shock strength of TB Eq. (26) is smaller than the TKdVB shock but, for the case of kappa distribution, TB shock is stronger than TKdVB shock. For the case of Cairns and Maxwellian distribution, the trend is similar to that of kappa but owing to the paucity of space, we have not shown the effect of increasing viscosity for Cairns and Maxwellian distribution.

V. SUMMARY AND CONCLUSION

We have obtained the expressions of number densities of trapped electrons for non-Maxwellian velocity distribution functions, namely, generalized (r, q), kappa and Cairns. It is well known that the electron trapping changes the nature of nonlinearity and, therefore, one obtains the modified nonlinear partial differential equation (NLPDE) as a consequence. To demonstrate the importance of incorporating trapping of electrons for non-Maxwellian distribution functions, we have derived nonlinear equations of ion sound excitations in an unmagnetized, homogeneous plasma with electron trapping both for the dispersive and dis-

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sipative cases in the weak nonlinearity limit. We have obtained trapped Korteweg-de Vries (TKdV), TKdV Burgers (TKdVB) and trapped Burgers (TB) equations for our physical model. It has been found that the flatness parameter r enhances the amplitude whereas the tail parameter q mitigates the amplitude of the IA solitons and shocks. It means that pumping in low energy electrons increases the amplitude whereas enhancing the percentage of high energy electrons enervates the amplitude of solitons and shocks under investigation. Interestingly, it has been found that the ranges of the flatness parameter r over which TKdVB and TB equations form are different which in a way give us the existence regimes of IA shock structures admitted by these two equations.

We have also made a comparison to highlight the differences among different non-Maxwellian distribution functions for electron trapping and it has been found that the shock strength for TKdVB equation is maximum for flat topped, intermediate for Cairns and minimum for kappa distribution of electrons whereas the TB shocks and TKdV solitons show a different behavior for different values of r. It has been shown that Cairns and flat-topped distributions both of which increase the percentage of low electrons enhance the amplitude of the IA solitons and shocks whereas the tail parameter q and the kappa distribution parameter κ, which are both responsible for altering the number of high energy electrons, mitigate the amplitude of the nonlinear structures under investigation. The comparison of non-Maxwellian trapped electrons with Maxwellian trapped electrons has also been made in this paper which shows that the non-Maxwellian distributions significantly alter the behavior of the IA solitons and shocks in comparison with their Maxwellian counterpart. Since both trapping and non-Maxwellian distribution functions are ubiquitous in space plasmas, the present investigation is expected to be considered as an important step forward as it lays the theoretical foundations of the problem of trapping in non-Maxwellian plasmas which will hopefully urge the data scientists to look for the nonlinear structures of such type which is a rather unexplored area of research to date.

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AUTHOR DECLARATIONS

Conflicts of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Figure Captions

Fig. 1: Variation in the profile of shock structure for the TKdVB Eq. (23) against (a) the index parameter r and for $q = 3$, $\sigma = 0.1$ and $\eta = 0.3$, and (b) the index parameter q and for $r = 0.6$, $\sigma = 0.1$ and $\eta = 0.3$.

Fig. 2: Comparison of the non-Maxwellian and Maxwellian shock structures for the TKdVB Eq. (23), where for Maxwellian case: $r = 0, q \to \infty$, kappa case: $r = 0, q = 3$, Cairns case: $r = 0, \alpha = 0.15$ and generalized (r, q) case: $r = 0.6, q = 3$. The other parameters are $\sigma = 0.3$ and $\eta = 0.3$.

Fig. 3: Variation of shock structure for the TB Eq. (26) against the index parameter r. The other parameters are $q = 3$, $\sigma = 0.1$ and $\eta = 0.3$.

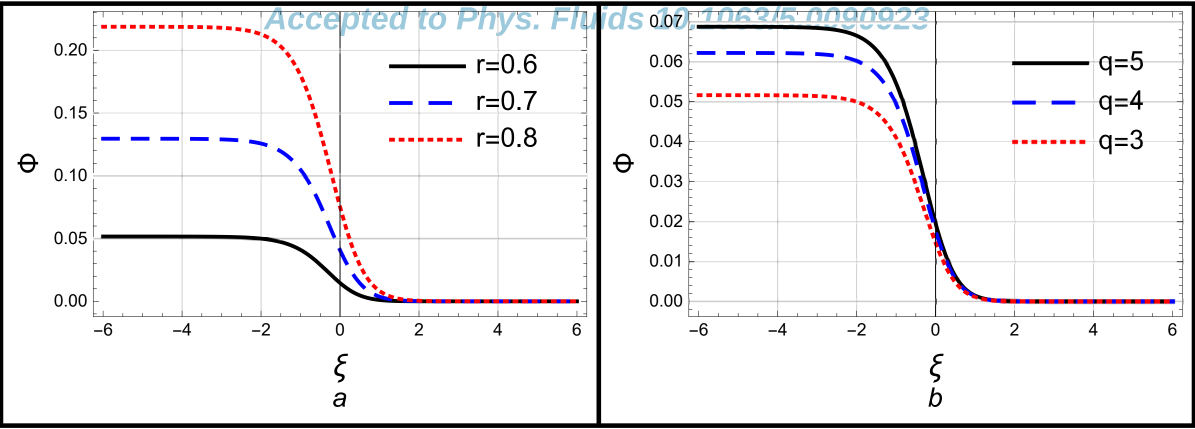
Fig. 4: Comparison of the non-Maxwellian and Maxwellian shock structures for the TB Eq. (26), for (a) Maxwellian case: $r = 0, q \to \infty$, kappa case: $r = 0, q = 3$, Cairns case: $r = 0, \alpha = 0.15$, and generalized (r, q) case: $r = 0.3, q = 3$, and (b) Maxwellian case: $r = 0, q \to \infty$, kappa case: $r = 0, q = 3$, Cairns case: $r = 0, \alpha = 0.15$ and generalized (r, q) case: $r = -0.05, q = 3$. The other parameters are $\sigma = 0.3$ and $\eta = 0.3$.

Fig. 5: Comparison of the non-Maxwellian and Maxwellian solitary structures for the TKdV Eq. (28), for (a) Maxwellian case: $r = 0, q \to \infty$, kappa case: $r = 0, q = 3$, Cairns

case: $r = 0, \alpha = 0.15$ and generalized (r, q) case: $r = 0.3, q = 3$ and (b) Maxwellian case: $r = 0, q \to \infty$, kappa case: $r = 0, q = 3$, Cairns case: $r = 0, \alpha = 0.15$ and generalized (r, q) case: $r = -0.05, q = 3$. The other parameters are $\sigma = 0.3$ and $\eta = 0.3$.

Fig. 6: Variation in shock structure for the TKdVB Eq. (23) against (a) η with $q = 3$ and $\sigma = 0.1$ and shows the generalized (r, q) distribution with $r = 0.6$ and (b) η with $q = 3$ and $\sigma = 0.1$ and shows the kappa distribution with $r = 0$.

Fig. 7: Variation in shock structure for the TB Eq. (26) against (a) η with $q = 3$ and shows the generalized (r, q) distribution with $r = 0.1$ and (b) η with $q = 3$ and shows the kappa distribution with $r = 0$.



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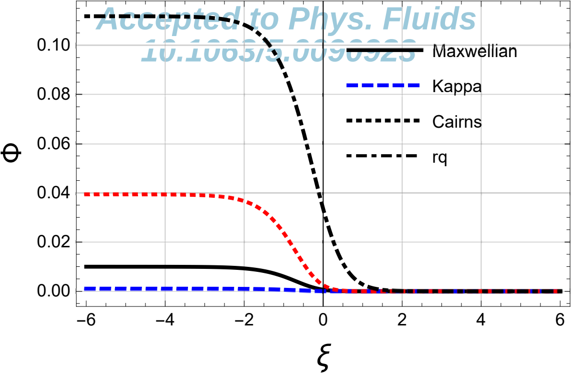
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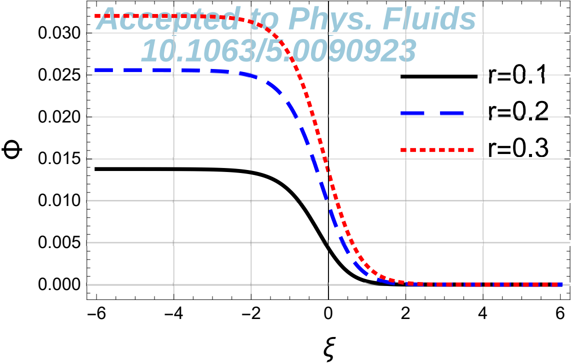
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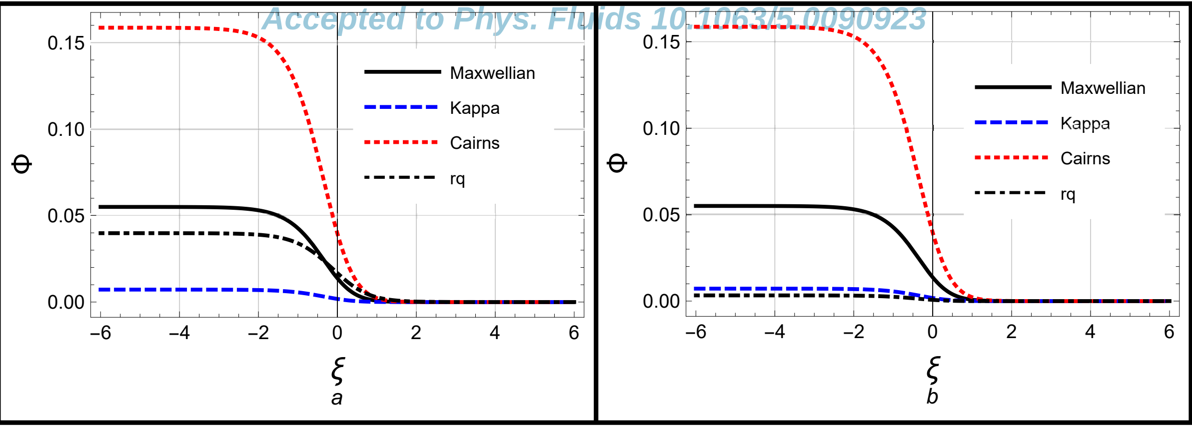
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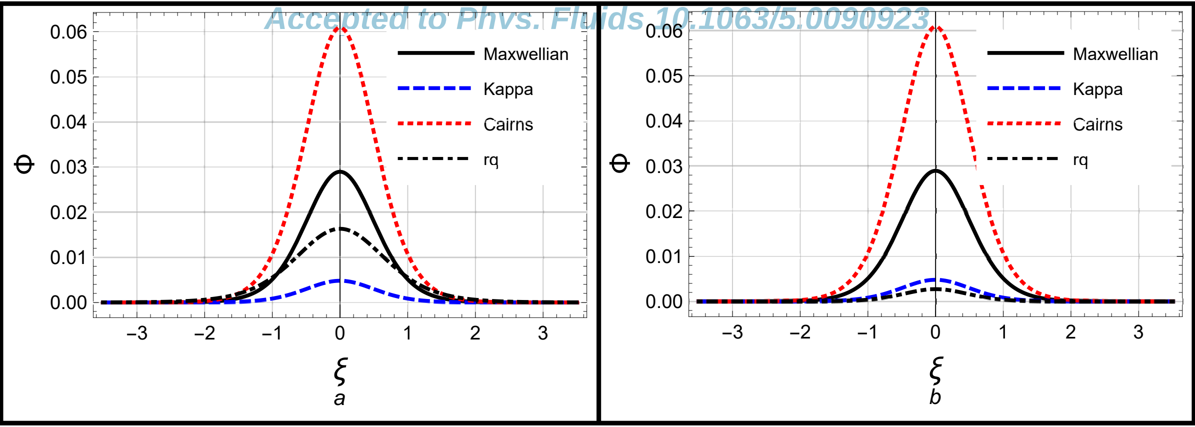

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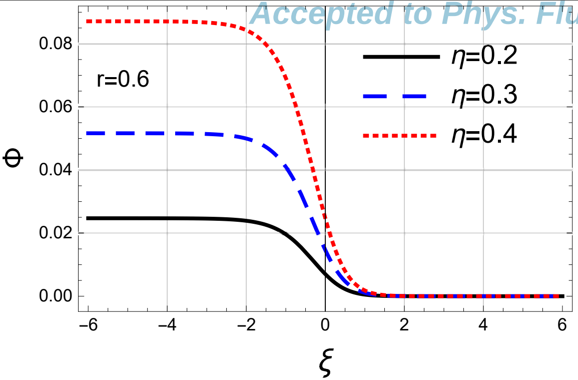
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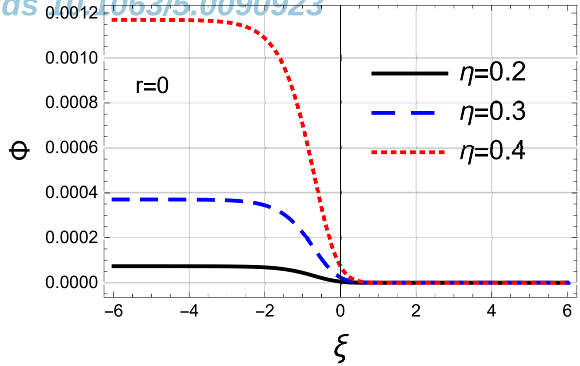
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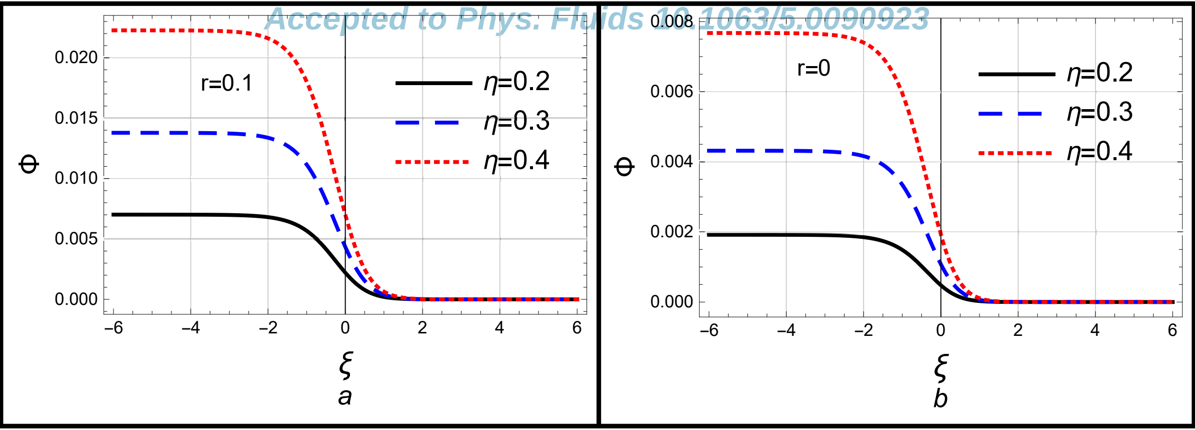
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