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# Monotone data modeling using rational functions

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**Abstract:** Rational schemes for shape preservation of monotone data both in 2D and 3D setups have been developed. $C^1$ rational cubic and partially blended bicubic functions are employed for this purpose. Monotonicity is achieved by extracting constraints on parameters involved in the description of these rational functions. Monotone curves and surfaces have been obtained, which provide evidence that the algorithm used fits most types of monotone data and produces visually pleasing results.

**Key words:** Monotone data, shape preservation, rational functions

## 1. Introduction

Construction of curves and surfaces in the digital age is at the heart of many scientific fields such as computer-aided geometric design (CAGD), computer graphics (CG), and computer-aided design (CAD). These applied fields integrate concepts from linear algebra, differential geometry, visualization, and numerical methods, all implemented in software. Curve and surface drawing techniques have given architects a cutting edge over the conventional drafting equipment and optimized the probability of turning their visions into practice, displaying them on computer screens in incredibly short spans of time. Three categories of data, i.e. monotonicity, positivity, and convexity, are often dominant in the majority of these applications. Rise in cholesterol level in the blood due to consuming food high in saturated fat, chlorofluorocarbon amounts in the depletion of the ozone layer, and uric acid levels in gout patients are some variables that always show monotonic pattern. Height and age of a person, number of computers in a computer laboratory, and population in a specific area are some real-life examples of positive data. Exponential and power functions, negative entropy, and area enclosed by a semicircle in the lower half plane are a few examples of curves that show convex behavior.

Mathematical models of the datasets exhibiting one of these three properties have ensured their significance in the fields of engineering, natural sciences, and social sciences. Construction of these models can assist in the study of effects of various constituents and in estimating the behavior of the system as a whole. For this purpose, the intrinsic shape of the data must be retained to avoid misinterpretation of the information. Standard methods of spline functions do not retain inherited characteristics of the data. Hence, by introducing some parameters into the spline structure, various characteristics of the data, including convexity, monotonicity, and positivity, can be conserved. Moreover, the number of parameters can be increased to pull the curve or surface towards the intrinsic shape of the data, keeping the smoothness of the results intact.

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Spline interpolating rational functions are very helpful in the envisioning of shaped data because they furnish smooth and captivating views of the data under consideration. Much notable work has been done previously to overcome this difficulty [1–10]. Kvasov [3] created an algorithm for interpolation by means of weighted cubic spline functions that keep the monotone and convex shape of discrete sets of data. Ibraheem et al. [4] evolved rational cubic and bicubic trigonometric methodologies to maintain the monotonicity of data in two and three dimensions. Constraints were attained on parameters of rational cubic and bicubic trigonometric functions. In [5], the authors did work on conservation of shaped data by evolving a $C^1$ rational cubic spline that formed a convex interpolant for given convex data. Sarfraz et al. [1] developed a piecewise rational cubic function of order $O(h^3)$ to visualize monotone data. Hussain et al. [9] designed a control point state of quadratic trigonometric functions that fulfill each of the properties of Bezier functions. Floater and Pena [7] considered and explained the types of monotonicity preservation of systems of bivariate function on a triangle. Sarfraz et al. [2] illustrated a rational cubic function having two parameters for the envisioning of positive-shaped data. The focus of the work was to present a smooth view of data. Sarfraz et al. [10] created a novel technique of curve interpolation. A piecewise rational cubic function encompassing two parameters was considered and their influence on the shape of the curve was scrutinized. Hussain et al. [6] extended the $GC^1$ quadratic trigonometric functions in [9] to $GC^1$ biquadratic trigonometric functions that confined four free parameters. Smooth and eye-catching monotone and positive surfaces were attained by developing limits on free parameters. The prime objective of this research is to acquire data-dependent constraints on parameters to retain monotonicity. For this purpose, rational cubic and bicubic functions by the authors in [2] are utilized. Monotonicity-preserving constraints for curve and surface data are developed in Section 3 and Section 4, respectively. Section 5 provides the implementation of a rational scheme developed on 2D and 3D datasets. The corresponding numerical results of values of derivatives and parameters are calculated using MATLAB and are shown in Section 6, followed by conclusions in Section 7.

## 2. Rational cubic and bicubic partially blended functions

Let us presume that $(\varepsilon_i, f_i)$ , $0 \leq i \leq n$, is the dataset supposed to be delineated on the interval $[a, b]$ in such a way that $a = \varepsilon_0 < \varepsilon_1 < \varepsilon_3 < \cdots < \varepsilon_n = b$. A rational cubic piecewise defined function possessing two free parameters defined in all the subintervals $I_i = [\varepsilon_i, \varepsilon_{i+1}], 0 \leq i \leq n-1$, is provided below:

$$S_i(\varepsilon) = \frac{p_i(\Phi)}{q_i(\Phi)}, \tag{1}$$

where

$$\begin{aligned} p_i(\Phi) &= \delta_i f_i (1-\Phi)^3 + \{\delta_i f_i + h_i \delta_i \dot{d}_i + 2 f_i\} \Phi (1-\Phi)^2 \\ &\quad + \{\xi_i f_{i+1} - h_i \xi_i \dot{d}_{i+1} + 2 f_{i+1}\} \Phi^2 (1-\Phi) + \xi_i f_{i+1} \Phi^3, \\ q_i(\Phi) &= \delta_i (1-\Phi)^2 + 2\Phi(1-\Phi) + \xi_i \Phi^2, \end{aligned}$$

and $\Phi = \frac{\varepsilon - \varepsilon_i}{h_i}, h_i = \varepsilon_{i+1} - \varepsilon_i$. The function provided in Eq. (1) holds $C^1$ continuity if the following attributes are satisfied:

$$\begin{aligned} S_i(\varepsilon_i) = f_i, S_i(\varepsilon_{i+1}) = f_{i+1}, \\ S_i^{(1)}(\varepsilon_i) = \dot{d}_i, S_i^{(1)}(\varepsilon_{i+1}) = \dot{d}_{i+1}. \end{aligned} \tag{2}$$

In Eq. (2), $S^{(1)}$ expresses the derivation with respect to $\varepsilon$ whereas $\dot{d}_i$ symbolizes the estimated values of derivatives at knots $\varepsilon_i$. The estimated derivative values may be already provided or estimated using some appropriate strategy. In this paper, derivatives are calculated using the geometric mean method detailed below:

$$\dot{d}_i = \begin{cases} 0 & \ddot{\Delta}_{i-1} = 0, \ddot{\Delta}_i = 0, \\ \ddot{\Delta}_{i-1}^{\frac{h_i}{h_{i-1}+h_i}} \ddot{\Delta}_i^{\frac{h_{i-1}}{h_{i-1}+h_i}} & i = 1, 2, \ldots, n-1. \end{cases}$$

$$\dot{d}_1 = \begin{cases} 0 & \ddot{\Delta}_1 = 0, \ddot{\Delta}_{3,1} = 0, \\ \ddot{\Delta}_1 [\frac{\ddot{\Delta}_1}{\ddot{\Delta}_{3,1}}]^{\frac{h_1}{h_2}} & otherwise. \end{cases}$$

$$\dot{d}_n = \begin{cases} 0 & \ddot{\Delta}_{n-1} = 0, \ddot{\Delta}_{n,n-2} = 0, \\ \ddot{\Delta}_{n-1} [\frac{\ddot{\Delta}_{n-1}}{\ddot{\Delta}_{n,n-2}}]^{\frac{h_{n-1}}{h_{n-2}}} & otherwise. \end{cases}$$

Here, $\ddot{\Delta}_i = \frac{f_{i+1}-f_i}{h_i}, i = 0, 1, 2, \ldots, n-1$ and $\ddot{\Delta}_{3,1} = \frac{f_3-f_1}{\varepsilon_3-\varepsilon_1}$, $\ddot{\Delta}_{n,n-2} = \frac{f_n-f_{n-2}}{\varepsilon_n-\varepsilon_{n-2}}$.
Similarly, for 3D data these are defined as follows:

$$\breve{F}^{\varepsilon}_{i,j} = \begin{cases} 0 & \ddot{\Delta}_{i-1,j} = 0, \ddot{\Delta}_{i,j} = 0, \\ \ddot{\Delta}_{i-1,j}^{\frac{h_i}{h_{i-1}+h_i}} \ddot{\Delta}_{i,j}^{\frac{h_{i-1}}{h_{i-1}+h_i}} & otherwise. \end{cases}$$

$$\breve{F}^{\varepsilon}_{1,j} = \begin{cases} 0 & \ddot{\Delta}_{1,j} = 0, \ddot{\Delta}_{31,j} = 0, \\ \ddot{\Delta}_{1,j} [\frac{\ddot{\Delta}_{1,j}}{\ddot{\Delta}_{31,j}}]^{\frac{h_1}{h_2}} & otherwise. \end{cases}$$

$$\breve{F}^{\varepsilon}_{n,j} = \begin{cases} 0 & \ddot{\Delta}_{n-1,j} = 0, \ddot{\Delta}_{n(n-2),j} = 0, \\ \ddot{\Delta}_{n-1,j} [\frac{\ddot{\Delta}_{n-1,j}}{\ddot{\Delta}_{n(n-2),j}}]^{\frac{h_{n-1}}{h_{n-2}}} & otherwise. \end{cases}$$

$$\breve{F}^{f}_{i,j} = \begin{cases} 0 & if \breve{\Delta}_{i,j-1} = 0, \breve{\Delta}_{i,j} = 0, \\ \breve{\Delta}_{i,j-1}^{\frac{\breve{h}_j}{\breve{h}_{j-1}+\breve{h}_j}} \breve{\Delta}_{i,j}^{\frac{\breve{h}_{j-1}}{\breve{h}_{j-1}+\breve{h}_j}} & otherwise \end{cases}$$

$$\breve{F}^{f}_{i,1} = \begin{cases} 0 & \breve{\Delta}_{i,1} = 0, \breve{\Delta}_{i,31} = 0, \\ \breve{\Delta}_{i,1} [\frac{\breve{\Delta}_{i,1}}{\breve{\Delta}_{i,31}}]^{\frac{\breve{h}_1}{\breve{h}_2}} & otherwise. \end{cases}$$

$$\breve{F}^{f}_{i,m} = \begin{cases} 0 & \breve{\Delta}_{i,m} = 0, \breve{\Delta}_{i,m(m-2)} = 0, \\ \breve{\Delta}_{i,m} [\frac{\breve{\Delta}_{i,m}}{\breve{\Delta}_{i,m(m-2)}}]^{\frac{\breve{h}_{m-1}}{\breve{h}_{m-2}}} & otherwise. \end{cases}$$

Here, $\ddot{\Delta}_{31,j} = \frac{\breve{F}_{3,j}-\breve{F}_{1,j}}{\varepsilon_3-\varepsilon_1}$, $\ddot{\Delta}_{n(n-2),j} = \frac{\breve{F}_{n,j}-\breve{F}_{n-2,j}}{\varepsilon_n-\varepsilon_{n-2}}$, $\breve{\Delta}_{i,31} = \frac{\breve{F}_{i,3}-\breve{F}_{i,1}}{f_3-f_1}$, $\breve{\Delta}_{i,m(m-2)} = \frac{\breve{F}_{i,m}-\breve{F}_{i,m-2}}{f_m-f_{m-2}}$,
$\ddot{\Delta}_{i,j} = \frac{\breve{F}_{i+1,j}-\breve{F}_{i,j}}{h_i}$, $\breve{\Delta}_{i,j} = \frac{\breve{F}_{i,j+1}-\breve{F}_{i,j}}{\breve{h}_j}$. $\forall i, j$.

Here, a fact worth mentioning is that for $\delta_i = \xi_i = 1$ in the interval $[\varepsilon_i, \varepsilon_{i+1}]$, the rational cubic function in Eq. (1) alters to the basic cubic Hermite spline. The rational cubic piecewise function supplied in Eq. (1) is advanced to a rational bicubic function delineated over a given set of 3D data points $(\varepsilon_i, f_j, \breve{F}_{i,j})$ where $0 \le i \le n$ and $0 \le j \le m$ on rectangular mesh $\breve{D} = [\varepsilon_0, \varepsilon_n] \times [f_0, f_m]$. Let $\hat{\rho}$: $\dot{a} = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_n = \dot{b}$ be the partitioning of $[\dot{a}, \dot{b}]$ and $\breve{\rho}$: $\dot{c} = f_0 < f_1 < f_2 < f_3 < \ldots < f_m = \dot{d}$ be the partitioning of $[\dot{c}, \dot{d}]$. We elucidate a bicubic partially blended rational function defined over rectangular patch $[\varepsilon_i, \varepsilon_{i+1}] \times [f_j, f_{j+1}]$ as:

$$S(\varepsilon, f) = -ABC^T. \tag{3}$$

Here,

$$B = \begin{bmatrix} 0 & S(\varepsilon, f_j) & S(\varepsilon, f_{j+1}) \\ S(\varepsilon_i, f) & S(\varepsilon_i, f_j) & S(\varepsilon_i, f_{j+1}) \\ S(\varepsilon_{i+1}, f) & S(\varepsilon_{i+1}, f_j) & S(\varepsilon_{i+1}, f_{j+1}) \end{bmatrix}$$

and $A$ and $C$ are row matrices given as:

$$A = \begin{bmatrix} -1 & a_0(\Phi) & a_1(\Phi) \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & c_0(\Psi) & c_1(\Psi) \end{bmatrix},$$

where $a_0(\Phi) = (1-\Phi)^2(1+2\Phi), a_1(\Phi) = \Phi^2(3-2\Phi),\ c_0(\Psi) = (1-\Psi)^2(1+2\Psi), c_1(\Psi) = \Psi^2(3-2\Psi)$. Furthermore, $\Phi = \frac{(\varepsilon-\varepsilon_i)}{h_i}$ and $\Psi = \frac{(f-f_j)}{\breve{h}_j}$ and $0 \le \Phi \le 1$ and $0 \le \Psi \le 1$, $h_i = \varepsilon_{i+1} - \varepsilon_i, \breve{h}_j = f_{j+1} - f_j$. Four rational cubic functions $S(\varepsilon, f_j), S(\varepsilon, f_{j+1}), S(\varepsilon_i, f), S(\varepsilon_{i+1}, f)$ are acquired just like the one in Eq. (1) delineated on the boundary of rectangular patch $[\varepsilon_i, \varepsilon_{i+1}] \times [f_j, f_{j+1}]$ as:

$$S(\varepsilon, f_j) = \frac{J_0(1-\Phi)^3 + J_1\Phi(1-\Phi)^2 + J_2\Phi^2(1-\Phi) + J_3\Phi^3)}{q_1(\Phi)}, \tag{4}$$

and values corresponding to $J_i, i = 0,1,2,3$ are:
$J_0 = \delta_{i,j}\breve{F}_{i,j}, J_1 = \delta_{i,j}\breve{F}_{i,j} + h_i\delta_{i,j}\breve{F}^{\varepsilon}_{i,j} + 2\breve{F}_{i,j}$ , $J_2 = \xi_{i,j}\breve{F}_{i+1,j} - \xi_{i,j}\breve{F}^{\varepsilon}_{i+1,j}h_i + 2\breve{F}^{\varepsilon}_{i+1,j}$ , $J_3 = \xi_{i,j}\breve{F}_{i+1,j}$.
$q_1(\Phi) = \delta_{i,j}(1-\Phi)^2 + \Phi(1-\Phi) + \xi_{i,j}\Phi^2$.
In similar fashion,

$$S(\varepsilon, f_{j+1}) = \frac{K_0(1-\Phi)^3 + K_1\Phi(1-\Phi)^2 + K_2\Phi^2(1-\Phi) + K_3\Phi^3}{q_2(\Phi)}, \tag{5}$$

where the values of $K_i, i = 0,1,2,3$ are:
$K_0 = \delta_{i,j+1}\breve{F}_{i,j+1}, K_1 = \delta_{i,j+1}\breve{F}_{i,j+1} + h_i\delta_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1} + 2\breve{F}_{i,j+1}$,
$K_2 = \xi_{i,j+1}\breve{F}_{i+1,j+1} - \xi_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1}h_i + 2\breve{F}_{i+1,j+1}, K_3 = \xi_{i,j+1}\breve{F}_{i+1,j+1}$,
$q_2(\Phi) = \delta_{i,j+1}(1-\Phi)^2 + \Phi(1-\Phi) + \xi_{i,j+1}\Phi^2$.
Likewise,

$$S(\varepsilon_i, f) = \frac{L_0(1-\Psi)^3 + L_1\Psi(1-\Psi)^2 + L_2\Psi^2(1-\Psi) + L_3\Psi^3)}{q_3(\Psi)}, \tag{6}$$

and values corresponding to $L_i, i = 0,1,2,3$ are:

$L_0 = \bar{\delta}_{i,j}\breve{F}_{i,j}, L_1 = \bar{\delta}_{i,j}\breve{F}_{i,j} + \breve{h}_j\bar{\delta}_{i,j+1}\breve{F}^f_{i,j} + 2\breve{F}_{i,j}$ , $L_2 = \bar{\xi}_{i,j}\breve{F}_{i,j+1} - \bar{\xi}_{i,j}\breve{F}^f_{i,j+1}\breve{h}_j + 2\breve{F}_{i,j+1}, L_3 = \bar{\xi}_{i,j}\breve{F}_{i,j+1}$ ,
$q_3(\Psi) = \bar{\delta}_{i,j+1}(1-\Psi)^2 + \Psi(1-\Psi) + \bar{\xi}_{i,j+1}\Psi^2$.
In like manner,

$$S(\varepsilon_{i+1}, f) = \frac{M_0(1-\Psi)^3 + M_1\Psi(1-\Psi)^2 + M_2\Psi^2(1-\Psi) + M_3\Psi^3}{q_4(\Psi)}, \tag{7}$$

where the values corresponding to $M_i, i = 0,1,2,3$ are:
$M_0 = \bar{\delta}_{i+1,j}\breve{F}_{i+1,j}, M_1 = \bar{\delta}_{i+1,j}\breve{F}_{i+1,j} + \breve{h}_j\bar{\delta}_{i+1,j}\breve{F}^f_{i+1,j} + 2\breve{F}_{i+1,j}$ ,
$M_2 = \bar{\xi}_{i+1,j}\breve{F}_{i+1,j+1} - \bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j+1}\breve{h}_j + 2\breve{F}_{i+1,j+1}, M_3 = \bar{\xi}_{i+1,j}\breve{F}_{i+1,j+1}$,
$q_4(\Psi) = \bar{\delta}_{i+1,j}(1-\Psi)^2 + \Psi(1-\Psi) + \bar{\xi}_{i+1,j}\Psi^2$.

## 3. Monotone curve model

Assume $(\varepsilon_i, f_i)$ to be the provided dataset where $0 \le i \le n$ . Suppose $f_i, i = 0,1,2,\ldots,n$ is a monotonic dataset, i.e. $f_i \le f_{i+1}$. Also,
$h_i = \varepsilon_{i+1} - \varepsilon_i$ , $\ddot{\Delta}_i = \frac{f_{i+1}-f_i}{h_i} \ge 0$.
Further, let us assume that $\dot{d}_i, 0 \le i \le n$, symbolizes the estimated derivative values at the points $\varepsilon_i, 0 \le i \le n$. Also, the vital monotonicity condition $\dot{d}_i \ge 0, 1 \le i \le n$ is satisfied. The free parameters within the interval $[\varepsilon_i, \varepsilon_{i+1}]$ are $\delta_i$ and $\xi_i$. The piecewise rational cubic function [2] is monotonically increasing if and only if $S^{(1)}_i(\varepsilon) \ge 0 \forall \varepsilon \in [\varepsilon_i, \varepsilon_{i+1}]$.

$$S^{(1)}_i(\varepsilon) = \frac{\ddot{a}_1(1-\Phi)^4 + \ddot{a}_2\Phi(1-\Phi)^3 + \ddot{a}_3\Phi^2(1-\Phi)^2 + \ddot{a}_4\Phi^3(1-\Phi) + \ddot{a}_5\Phi^4}{[q_i(\Phi)]^2}, \tag{8}$$

where $\ddot{a}_i, 1 \le i \le 5$ are given as:
$\ddot{a}_1 = \delta_i^2\dot{d}_i, \ddot{a}_2 = 2\delta_i\xi_i\ddot{\Delta}_i + 4\delta_i\ddot{\Delta}_i - 2\delta_i\xi_i\dot{d}_{i+1}$ , $\ddot{a}_3 = 4\delta_i\xi_i\ddot{\Delta}_i + 2\delta_i\ddot{\Delta}_i + 2\xi_i\ddot{\Delta}_i + 4\ddot{\Delta}_i - \delta_i\xi_i\dot{d}_{i+1} - 2\xi_i\dot{d}_{i+1} - 2\delta_i\dot{d}_i - \delta_i\xi_i\dot{d}_i$ ,
$\ddot{a}_4 = 2\delta_i\xi_i\ddot{\Delta}_i + 4\xi_i\ddot{\Delta}_i - 2\delta_i\xi_i\dot{d}_i, \ddot{a}_5 = \xi_i^2\dot{d}_{i+1}$.
Now $S^{(1)}_i(\varepsilon) \ge 0$, iff $\ddot{a}_j > 0, 1 \le j \le 5$ in all of the subintervals $[\varepsilon_i, \varepsilon_{i+1}]$. Consequently, the following constraints have been derived on free parameters:
$\xi_i > \frac{2\ddot{\Delta}_i}{\dot{d}_{i+1} - \ddot{\Delta}_i}, \delta_i > \frac{2\ddot{\Delta}_i}{\dot{d}_i - \ddot{\Delta}_i}, \xi_i > \frac{2(\dot{d}_i - \ddot{\Delta}_i)}{4\ddot{\Delta}_i - \dot{d}_i - \dot{d}_{i+1}}$.
The above discussion can be summarized as follows:

**Theorem 1** *The rational cubic piecewise function in [2] retains monotonicity if in each subinterval* $[\varepsilon_i, \varepsilon_{i+1}], 0 \le i \le n-1, \delta_i$ *and* $\xi_i$ *fulfill:*

$$\delta_i = \tilde{\lambda}_i + \frac{2\ddot{\Delta}_i}{\dot{d}_i - \ddot{\Delta}_i}, \tilde{\lambda}_i > 0,$$

$$\xi_i = \tilde{\sigma}_i + \max\{0, \frac{2\ddot{\Delta}_i}{\dot{d}_{i+1} - \ddot{\Delta}_i}, \frac{2(\dot{d}_i - \ddot{\Delta}_i)}{4\ddot{\Delta}_i - \dot{d}_i - \dot{d}_{i+1}}\}, \tilde{\sigma}_i > 0.$$

## 4. Monotone surface model

Assume the given set of 3D data points $(\varepsilon_i, f_j, \breve{F}_{i,j})$ where $0 \leq i \leq n$ and $0 \leq j \leq m$ on rectangular mesh $\breve{D} = [\varepsilon_0, \varepsilon_n] \times [f_0, f_m]$ such that the following conditions are met:

$\breve{F}_{i,j} < \breve{F}_{i+1,j}\,,\ \breve{F}_{i,j} < \breve{F}_{i,j+1}\,,\ \breve{F}^{\varepsilon}_{i,j} > 0\,,\ \breve{F}^{f}_{i,j} > 0,$ and $\ddot{\Delta}_{i,j} > 0\,,\ \breve{\Delta}_{i,j} > 0,$

where $\ddot{\Delta}_{i,j} = \frac{\breve{F}_{i+1,j} - \breve{F}_{i,j}}{h_i}\,,\ \breve{\Delta}_{i,j} = \frac{\breve{F}_{i,j+1} - \breve{F}_{i,j}}{\breve{h}_j}\,.$

For the surface patch in [2] to be monotone, we only need to establish that the boundary curves $S(\varepsilon, f_j), S(\varepsilon, f_{j+1}), S(\varepsilon_i, f), \text{and} S(\varepsilon_{i+1}, f)$ are all monotone. First consider $S(\varepsilon, f_j)$. It is known that $S(\varepsilon, f_j)$ is surely monotone if $S^{(1)}_i(\varepsilon, f_j) > 0$.

$$S^{(1)}_i(\varepsilon, f_j) = \frac{A_0(1-\Phi)^4 + A_1\Phi(1-\Phi)^3 + A_2\Phi^2(1-\Phi)^2 + A_3\Phi^3(1-\Phi) + A_4\Phi^4)}{[q_1(\Phi)]^2}. \tag{9}$$

The values corresponding to $A_i, i = 0,1,2,3,4$ are:

$A_0 = \delta^2_{i,j}\breve{F}^{\varepsilon}_{i,j},\ A_1 = 2\delta_{i,j}\xi_{i,j}\ddot{\Delta}_{i,j} + 4\delta_{i,j}\ddot{\Delta}_{i,j} - 2\delta_{i,j}\xi_{i,j}\breve{F}_{i+1,j},$

$A_2 = 4\delta_{i,j}\xi_{i,j}\ddot{\Delta}_{i,j} + 2\delta_{i,j}\ddot{\Delta}_{i,j} + 2\xi_{i,j}\ddot{\Delta}_{i,j} + 4\ddot{\Delta}_{i,j} - \delta_{i,j}\xi_{i,j}\breve{F}^{\varepsilon}_{i+1,j} - 2\xi_{i,j}\breve{F}^{\varepsilon}_{i+1,j} - 2\delta_{i,j}\breve{F}^{\varepsilon}_{i,j} - \delta_{i,j}\xi_{i,j}\breve{F}^{\varepsilon}_{i,j},$

$A_3 = 2\delta_{i,j}\xi_{i,j}\ddot{\Delta}_{i,j} + 4\xi_{i,j}\ddot{\Delta}_{i,j} - 2\delta_{i,j}\xi_{i,j}\breve{F}^{\varepsilon}_{i,j}\ A_4 = \xi^2_{i,j}\breve{F}^{\varepsilon}_{i+1,j}\,.$

When $S^{(1)}_i(\varepsilon, f_j) > 0$, all $A_i, i = 1,2,3$, are necessarily positive. Thus, $A_1 > 0, A_2 > 0$, and $A_3 > 0$ yield the following constraints on $\delta_{i,j}$ and $\xi_{i,j}$:

$\delta_{i,j} > \max\{0, \frac{2\ddot{\Delta}_{i,j}}{\breve{F}^{\varepsilon}_{i+1,j} - \ddot{\Delta}_{i,j}}, \frac{2(\breve{F}^{\varepsilon}_{i,j} - \ddot{\Delta}_{i,j})}{4\ddot{\Delta}_{i,j} - \breve{F}^{\varepsilon}_{i,j} - \breve{F}^{\varepsilon}_{i+1,j}}\}, \xi_{i,j} > \max\{0, \frac{2\ddot{\Delta}_{i,j}}{\breve{F}^{\varepsilon}_{i,j} - \ddot{\Delta}_{i,j}}\}$. Continuing in a similar fashion, $S(\varepsilon, f_{j+1})$ is said to be monotone if $S^{(1)}_i(\varepsilon, f_{j+1}) > 0$.

$$S^{(1)}_i(\varepsilon, f_{j+1}) = \frac{B_0(1-\Phi)^4 + B_1\Phi(1-\Phi)^3 + B_2\Phi^2(1-\Phi)^2 + B_3\Phi^3(1-\Phi) + B_4\Phi^4)}{[q_2(\Phi)]^2}. \tag{10}$$

Here,

$B_0 = \delta^2_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1},\ B_1 = 2\delta_{i,j+1}\xi_{i,j+1}\ddot{\Delta}_{i,j+1} + 4\delta_{i,j+1}\ddot{\Delta}_{i,j+1} - 2\delta_{i,j+1}\xi_{i,j+1}\breve{F}^{\varepsilon}_{i+1,j+1},$

$B_2 = 4\delta_{i,j+1}\xi_{i,j+1}\ddot{\Delta}_{i,j+1} + 2\delta_{i,j+1}\ddot{\Delta}_{i,j+1} + 2\xi_{i,j+1}\ddot{\Delta}_{i,j+1} + 4\ddot{\Delta}_{i,j+1} - \delta_{i,j+1}\xi_{i,j+1}\breve{F}^{\varepsilon}_{i+1,j+1} - 2\xi_{i,j+1}\breve{F}^{\varepsilon}_{i+1,j+1} - 2\delta_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1} - \delta_{i,j+1}\xi_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1}\,,$

$B_3 = 2\delta_{i,j+1}\xi_{i,j+1}\ddot{\Delta}_{i,j+1} + 4\xi_{i,j+1}\ddot{\Delta}_{i,j+1} - 2\delta_{i,j+1}\xi_{i,j+1}\breve{F}^{\varepsilon}_{i,j+1},\ B_4 = \xi^2_{i,j+1}\breve{F}^{\varepsilon}_{i+1,j+1}\,.$

$S^{(1)}_i(\varepsilon, f_{j+1}) > 0$ where $B_i, i = 0,1,2,3,4$ are essentially positive. Thus, $B_1 > 0, B_2 > 0$, and $B_3 > 0$ yield the following constraints on $\delta_{i,j+1}$ and $\xi_{i,j+1}$:

$\xi_{i,j+1} > \max\{0, \frac{2\ddot{\Delta}_{i,j+1}}{\breve{F}^{\varepsilon}_{i+1,j+1} - \ddot{\Delta}_{i,j+1}}, 2\frac{(\breve{F}^{\varepsilon}_{i,j+1} - \ddot{\Delta}_{i,j+1})}{4\ddot{\Delta}_{i,j+1} - \breve{F}^{\varepsilon}_{i,j+1} - \breve{F}^{\varepsilon}_{i+1,j+1}}\}\,, \delta_{i,j+1} > \max\{0, \frac{2\ddot{\Delta}_{i,j+1}}{\breve{F}^{\varepsilon}_{i,j+1} - \ddot{\Delta}_{i,j+1}}\}\,.$

Likewise, $S(\varepsilon_i, f)$ is monotone if $S^{(1)}_i(\varepsilon_i, f) > 0$.

$$S^{(1)}_i(\varepsilon_i, f) = \frac{C_0(1-\Psi)^4 + C_1\Psi(1-\Psi)^3 + C_2\Psi^2(1-\Psi)^2 + C_3\Psi^3(1-\Psi) + C_4\Psi^4)}{[q_3(\Psi)]^2}, \tag{11}$$

where

$C_0 = \bar{\delta}^2_{i,j}\breve{F}^{f}_{i,j},\ C_1 = 2\bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{\Delta}_{i,j} + 4\bar{\delta}_{i,j}\breve{\Delta}_{i,j} - 2\bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{F}^{f}_{i,j+1}\,,$

$C_2 = 4\bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{\Delta}_{i,j} + 2\bar{\delta}_{i,j}\breve{\Delta}_{i,j} + 2\bar{\delta}_{i,j}\breve{\Delta}_{i,j} + 4\breve{\Delta}_{i,j} - \bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{F}^f_{i,j+1} - 2\bar{\delta}_{i,j}\breve{F}^f_{i,j+1} - 2\bar{\delta}_{i,j}\breve{F}^f_{i,j} - \bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{F}^f_{i,j}$,
$C_3 = 2\bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{\Delta}_{i,j} + 4\bar{\xi}_{i,j}\breve{\Delta}_{i,j} - 2\bar{\delta}_{i,j}\bar{\xi}_{i,j}\breve{F}^f_{i,j}$, $C_4 = \bar{\xi}^2_{i,j}\breve{F}^f_{i,j+1}$.

Similarly, for $S^{(1)}_i(\varepsilon_i\,,\ f) > 0$, $C_i\,, i\ = 0, 1, 2, 3, 4$ should necessarily be positive. Here, $C_1 > 0, C_2 > 0$, and $C_3 > 0$ yield the following constraints on $\bar{\delta}_{i,j}$ and $\bar{\xi}_{i,j}$ :

$\bar{\delta}_{i,j} > \max\{0, \frac{2\breve{\Delta}_{i,j}}{\breve{F}^f_{i,j+1}\breve{\Delta}_{i,j}}, \frac{2(\breve{F}^f_{i,j}-\breve{\Delta}_{i,j})}{4\breve{\Delta}_{i,j}-\breve{F}^f_{i,j}-\breve{F}^f_{i,j+1}}\}$, $\bar{\xi}_{i,j} > \max\{0, \frac{(2\breve{\Delta}_{i,j})}{\breve{F}^f_{i,j}-\breve{\Delta}_{i,j}}\}$. Lastly, $S(\varepsilon_{i+1}, f)$ is considered to be monotone if $S^{(1)}_i(\varepsilon_{i+1}, f) > 0$.

$$S^{(1)}_i(\varepsilon_{i+1}, f) = \frac{D_0(1-\Psi)^4 + D_1\Psi(1-\Psi)^3 + D_2\Psi^2(1-\Psi)^2 + D_3\Psi^3(1-\Psi) + D_4\Psi^4}{[q_4(\Psi)]^2}. \quad (12)$$

Here we have:
$D_0 = \bar{\delta}^2_{i+1,j}\breve{F}^f_{i+1,j}$, $D_1 = 2\bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{\Delta}_{i+1,j} + 4\bar{\delta}_{i+1,j}\breve{\Delta}_{i+1,j} - 2\bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j+1}$,
$D_2 = 4\bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{\Delta}_{i+1,j} + 2\bar{\delta}_{i+1,j}\breve{\Delta}_{i+1,j} + 2\bar{\xi}_{i+1,j}\breve{\Delta}_{i+1,j} + 4\breve{\Delta}_{i+1,j} - \bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j+1} - 2\bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j+1} - 2\bar{\delta}_{i+1,j}\breve{F}^f_{i+1,j} - \bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j}$,
$D_3 = 2\bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{\Delta}_{i+1,j} + 4\bar{\xi}_{i+1,j}\breve{\Delta}_{i+1,j} - 2\bar{\delta}_{i+1,j}\bar{\xi}_{i+1,j}\breve{F}^f_{i+1,j}$, $D_4 = \bar{\xi}^2_{i+1,j}\breve{F}^f_{i+1,j}$.

Furthermore, for $S^{(1)}_i(\varepsilon_{i+1}, f) > 0$, $D_i, i = 0, 1, 2, 3, 4$ essentially is positive. From $D_1 > 0, D_2 > 0$, and $D_3 > 0$, we procure the following constraints on free parameters:

$\bar{\delta}_{i+1,j} > \max\{0, \frac{2\breve{\Delta}_{i+1,j}}{\breve{F}^f_{i+1,j+1}-\breve{\Delta}_{i+1,j}}, \frac{2(\breve{F}^f_{i+1,j}-\breve{\Delta}_{i+1,j})}{4\breve{\Delta}_{i+1,j}-\breve{F}^f_{i+1,j}-\breve{F}^f_{i+1,j+1}}\}$, $\bar{\xi}_{i+1,j} > \max\{0, \frac{2\breve{\Delta}_{i+1,j}}{\breve{F}^f_{i+1,j}-\breve{\Delta}_{i+1,j}}\}$. The following theorem summarize the whole of the above argument:

**Theorem 2** *The bicubic partially blended rational function supplied in [2] provides the envisaging of monotone data in the form of a monotone surface if in rectangular mesh $I_{ij} = [\varepsilon_i, \varepsilon_{i+1}] \times [f_j, f_{j+1}]$, the constraints provided as follows are satisfied by the free parameters:*

$\delta_{i,j} = c'_{i,j} + \max\{0, \frac{2\ddot{\Delta}_{i,j}}{\breve{F}^\varepsilon_{i+1,j}-\ddot{\Delta}_{i,j}}, \frac{2(\breve{F}^\varepsilon_{i,j}-\ddot{\Delta}_{i,j})}{4\ddot{\Delta}_{i,j}-\breve{F}^\varepsilon_{i,j}-\breve{F}^\varepsilon_{i+1,j}}\}, c'_{i,j} > 0;$

$\xi_{i,j} = d'_{i,j} + \max\{0, \frac{2\ddot{\Delta}_{i,j}}{\breve{F}^\varepsilon_{i,j}-\ddot{\Delta}_{i,j}}\}, d'_{i,j} > 0;$

$\delta_{i,j+1} = e'_{i,j} + \max\{0, \frac{2\ddot{\Delta}_{i,j+1}}{\breve{F}^\varepsilon_{i,j+1}-\ddot{\Delta}_{i,j+1}}\}, e'_{i,j} > 0;$

$\xi_{i,j+1} = g'_{i,j} + \max\{0, \frac{2\ddot{\Delta}_{i,j+1}}{\breve{F}^\varepsilon_{i+1,j+1}-\ddot{\Delta}_{i,j+1}}, 2\frac{(\breve{F}^\varepsilon_{i,j+1}-\ddot{\Delta}_{i,j+1})}{4\ddot{\Delta}_{i,j+1}-\breve{F}^\varepsilon_{i,j+1}-\breve{F}^\varepsilon_{i+1,j+1}}\}, g'_{i,j} > 0;$

$\bar{\delta}_{i,j} = j'_{i,j} + \max\{0, \frac{2\breve{\Delta}_{i,j}}{\breve{F}^f_{i,j+1}\breve{\Delta}_{i,j}}, \frac{2(\breve{F}^f_{i,j}-\breve{\Delta}_{i,j})}{4\breve{\Delta}_{i,j}-\breve{F}^f_{i,j}-\breve{F}^f_{i,j+1}}\}, j'_{i,j} > 0;$

$\bar{\xi}_{i,j} = k'_{i,j} + \max\{0, \frac{(2\breve{\Delta}_{i,j})}{\breve{F}^f_{i,j}-\breve{\Delta}_{i,j}}\}, k'_{i,j} > 0;$

$\bar{\delta}_{i+1,j} = l'_{i,j} + \max\{0, \frac{2\breve{\Delta}_{i+1,j}}{\breve{F}^f_{i+1,j+1}-\breve{\Delta}_{i+1,j}}, \frac{2(\breve{F}^f_{i+1,j}-\breve{\Delta}_{i+1,j})}{4\breve{\Delta}_{i+1,j}-\breve{F}^f_{i+1,j}-\breve{F}^f_{i+1,j+1}}\}, l'_{i,j} > 0;$

$\bar{\xi}_{i+1,j} = m'_{i,j} + \max\{0, \frac{2\breve{\Delta}_{i+1,j}}{\breve{F}^f_{i+1,j}-\breve{\Delta}_{i+1,j}}\}, m'_{i,j} > 0.$

## 5. Numerical examples

Monotonicity-conserving rational schemes developed in Section 3 and Section 4 have been applied on 2D monotone datasets (Table 1 and Table 2) and 3D monotone datasets (Table 3 and Table 4), corresponding to which monotone curves and surfaces are obtained as shown in Figure 1, Figure 2, Figure 3, and Figure 4. The tz and fz views of the monotone surface generated from the dataset in Table 3 are demonstrated in Figure 5 and Figure 6, respectively. Similarly, Figure 7 and Figure 8 represent tz and fz views of the monotone surface produced from the dataset in Table4.

**Table 1**. A 2D monotone dataset, I.

| $t_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f_i$ | 0 | 5 | 9 | 13 | 17 | 20 | 21 |

**Table 2**. A 2D monotone dataset, II.

| $t_i$ | 2 | 3 | 6.5 | 7 | 7.5 |
|---|---|---|---|---|---|
| $f_i$ | 2 | 3 | 17 | 23 | 29 |

**Table 3**. A 3D monotonic dataset generated by the function $\breve{F}(\varepsilon, f) = \sqrt{\varepsilon + f + 0.005}$.

| $f/\varepsilon$ | 0.1 | 1.49 | 2.52 | 3.49 | 3.5 |
|---|---|---|---|---|---|
| 0.1 | 0.4528 | 1.2629 | 1.6202 | 1.8960 | 1.8987 |
| 1.49 | 1.2629 | 1.7277 | 2.0037 | 2.2327 | 2.2349 |
| 2.52 | 1.6202 | 2.0037 | 2.2461 | 2.4525 | 2.4546 |
| 3.49 | 1.8960 | 2.2327 | 2.4525 | 2.6429 | 2.6448 |
| 3.5 | 1.8987 | 2.2349 | 2.4546 | 2.6448 | 2.6467 |

**Table 4**. A 3D monotonic dataset generated by the function $\breve{F}(\varepsilon, f) = exp(\varepsilon^{0.05} + f^{0.07})$.

| $f/\varepsilon$ | 2.5 | 11.10 | 25.65 |
|---|---|---|---|
| 2.5 | 0.4528 | 1.2629 | 1.6202 |
| 11.10 | 1.2629 | 1.7277 | 2.0037 |
| 25.65 | 1.6202 | 2.0037 | 2.2461 |

## 6. Numerical values

The values of the parameters and the derivatives involved in the proposed algorithm have been calculated using MATLAB software and are presented in Table 5, Table 6, Table 7, and Table 8.

## 7. Conclusion

A $C^1$ rational cubic interpolating scheme put forward by the authors in [2] encompassing two parameters has been used for the modeling of monotone two- and three-dimensional data evolving as an outcome of certain

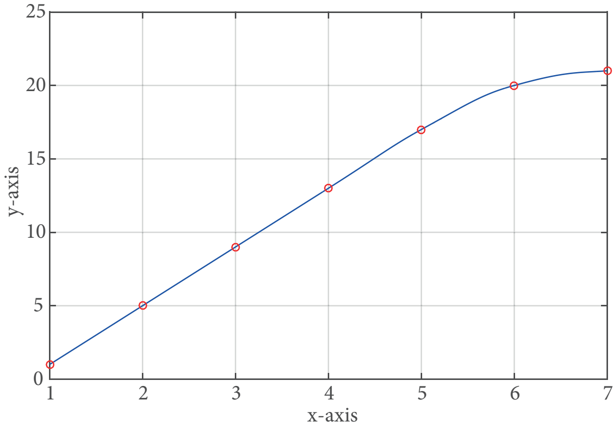


**Figure 1**. Monotonicity-conserving curve for dataset provided in Table 1.

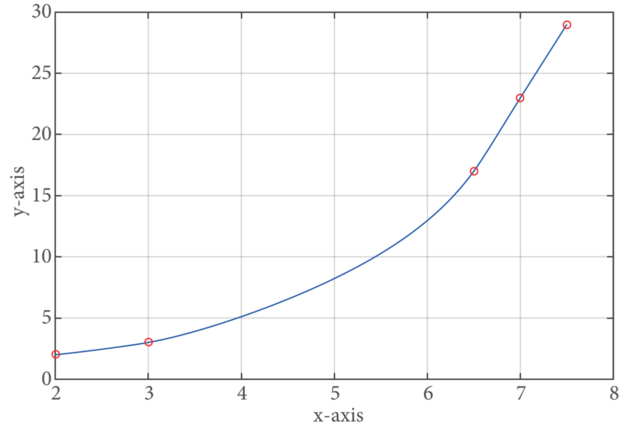


**Figure 2**. Monotonicity-conserving curve for dataset provided in Table 2.

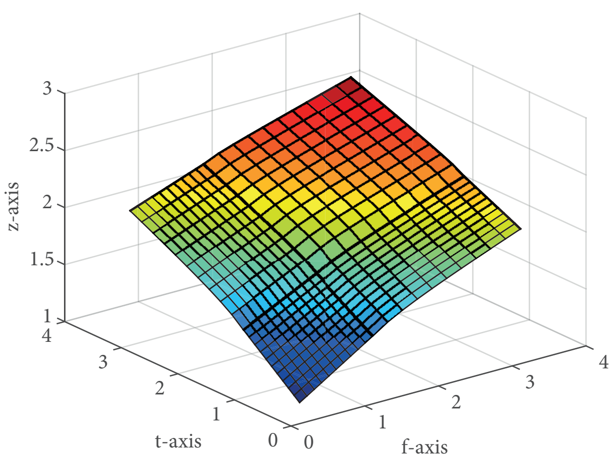


**Figure 3**. Monotone surface generated from dataset in Table 3.

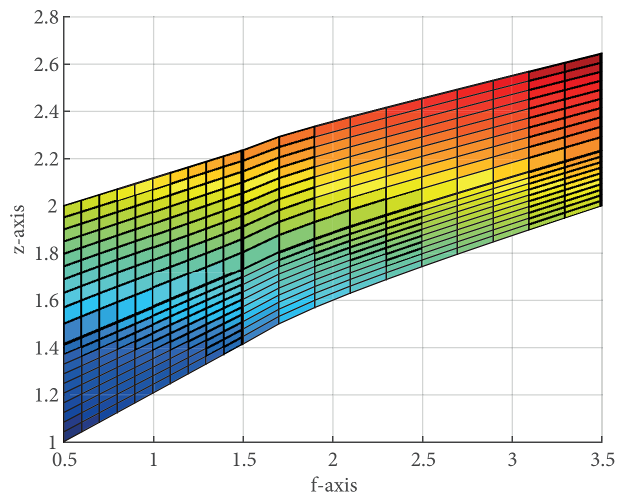


**Figure 4**. Monotone surface generated from dataset in Table 4.

**Table 5**. Numerical outcomes for Table 1.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\dot{d}_i$ | 4.0000 | 4.0000 | 4.0000 | 4.0000 | 3.4641 | 1.7321 | 0.5000 |
| $\delta_i$ | 3.2321 | 3.2321 | 3.2321 | 3.2321 | 3.2321 | 3.2321 | – |
| $\xi_i$ | 1.3281 | 1.3281 | 1.3281 | 1.3281 | 1.3281 | 1.3281 | – |

**Table 6**. Numerical outcomes for Table 2.

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\dot{d}_i$ | 0.70893 | 1.3608 | 10.46 | 12 | 12 |
| $\delta_i$ | 0.5 | 0.5 | 0.5 | 0.5 | – |
| $\xi_i$ | 0.5 | 0.5 | 0.5 | 0.5 | – |

**Table 7**. Numerical outcomes for Table 3.

| $(\xi_i, f_i)$ | 0.1 | 1.49 | 2.52 | 3.49 | 3.5 |
|---|---|---|---|---|---|
| Numerical outcomes for $\breve{F}^{\varepsilon}_{i,j}$ | | | | | |
| 0.1 | 0.2813 | 0.0762 | 0.0486 | 0.0357 | 0.0356 |
| 1.49 | 0.4326 | 0.2945 | 0.2518 | 0.2252 | 0.2250 |
| 2.52 | 0.3131 | 0.2510 | 0.2234 | 0.2044 | 0.2042 |
| 3.49 | 0.2637 | 0.2240 | 0.2039 | 0.1892 | 0.1891 |
| 3.5 | 0.2599 | 0.2204 | 0.2005 | 0.1859 | 0.1858 |
| Numerical outcomes for $\breve{F}^{f}_{i,j}$ | | | | | |
| 0.1 | 0.2813 | 0.4326 | 0.3131 | 0.2637 | 0.2599 |
| 1.49 | 0.0762 | 0.2945 | 0.2510 | 0.2240 | 0.2204 |
| 2.52 | 0.0486 | 0.2518 | 0.2234 | 0.2039 | 0.2005 |
| 3.49 | 0.0357 | 0.2252 | 0.2044 | 0.1892 | 0.1859 |
| 3.5 | 0.0356 | 0.2250 | 0.2042 | 0.1891 | 0.1858 |
| Numerical outcomes for $\delta_{i,j}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 1.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 2.52 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 3.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| Numerical outcomes for $\xi_{i,j}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 1.49 | 8.0909 | 20.25 | 28.506 | 36.173 | – |
| 2.52 | 19.79 | 31.5 | 40.085 | 48.141 | – |
| 3.49 | 2570.4 | 3687.5 | 4513.7 | 5291.1 | – |
| 3.5 | – | – | – | – | – |
| Numerical outcomes for $\delta_{i,j+1}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | |
| 1.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 2.52 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 3.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 3.5 | – | – | – | – | – |
| Numerical outcomes for $\xi_{i,j+1}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 1.49 | 20.25 | 28.506 | 36.173 | 36.251 | – |
| 2.52 | 31.5 | 40.085 | 48.141 | 48.224 | – |
| 3.49 | 3687.5 | 4513.7 | 5291.1 | 5299.1 | – |
| 3.5 | – | – | – | – | – |
| Numerical outcomes for $\breve{\delta}_{i,j}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 1.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 2.52 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 3.49 | 0.001 | 0.001 | 0.001 | 0.001 | – |
| 3.5 | – | – | – | – | – |
| | | | | | |
| Numerical outcomes for $\breve{\xi}_{i,j}$ | | | | | |
| 0.1 | 0.001 | 8.0909 | 19.79 | 2570.4 | – |
| 1.49 | 0.001 | 20.25 | 31.5 | 3687.5 | – |
| 2.52 | 0.001 | 28.506 | 40.085 | 4513.7 | – |
| 3.49 | 0.001 | 36.173 | 48.141 | 5291.1 | – |
| 3.5 | – | – | – | – | – |
| Numerical outcomes for $\breve{\delta}_{i+1,j}$ | | | | | |
| 0.1 | 0.001 | 0.001 | 0.001 | 0.001 | |
| 1.49 | 0.001 | 0.001 | 0.001 | 0.001 | |
| 2.52 | 0.001 | 0.001 | 0.001 | 0.001 | |
| 3.49 | 0.001 | 0.001 | 0.001 | 0.001 | |
| 3.5 | | | | | |
| Numerical outcomes for $\breve{\xi}_{i+1,j}$ | | | | | |
| 0.1 | 0.001 | 8.0909 | 19.79 | 2570.4 | – |
| 1.49 | 0.001 | 20.25 | 31.5 | 3687.5 | – |
| 2.52 | 0.001 | 28.506 | 40.085 | 4513.7 | – |
| 3.49 | 0.001 | 36.173 | 48.141 | 5291.1 | – |
| 3.5 | – | – | – | – | – |

**Table 8**. Numerical outcomes for Table 4.

| $(\xi_i, f_i)$ | 2.5 | 1.10 | 25.65 |
|---|---|---|---|
| Numerical outcomes for $\breve{F}^{xi}_{i,j}$ | | | |
| 2.5 | 0.018402 | 0.022177 | 0.024848 |
| 11.10 | 0.056414 | 0.063433 | 0.068132 |
| 25.65 | 0.0000829 | 0.00011379 | 0.00013793 |
| Numerical outcomes for $\dot{F}^{f}_{i,j}$ | | | |
| 2.5 | 0.034137 | 0.08483 | 0.00027203 |
| 11.10 | 0.038833 | 0.091988 | 0.00033832 |
| 25.65 | 0.041931 | 0.096534 | 0.00038523 |
| Numerical outcomes for $\delta_{i,j}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | 0.001 | 0.001 | – |
| 25.65 | – | – | – |
| Numerical outcomes for $\xi_{i,j}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | 2.3508 | 2.3508 | – |
| 25.65 | – | – | – |
| Numerical outcomes for $\delta_{i,j+1}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | | 0.001 | – |
| 25.65 | 0.001 | – | – |
| Numerical outcomes for $\xi_{i,j+1}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | 2.3508 | 2.3508 | – |
| 25.65 | – | – | – |
| Numerical outcomes for $\bar{\delta}_{i,j}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | 0.001 | 0.001 | – |
| 25.65 | – | – | – |
| Numerical outcomes for $\bar{\xi}_{i,j}$ | | | |
| 2.5 | 0.001 | 2.53 | – |
| 11.10 | 0.001 | 2.53 | – |
| 25.65 | | – | – |
| Numerical outcomes for $\bar{\delta}_{i+1,j}$ | | | |
| 2.5 | 0.001 | 0.001 | – |
| 11.10 | 0.001 | 0.001 | – |
| 25.65 | – | – | – |
| Numerical outcomes for $\bar{\xi}_{i+1,j}$ | | | |
| 2.5 | 0.001 | 2.53 | – |
| 11.10 | 0.001 | 2.53 | – |
| 25.65 | – | – | – |

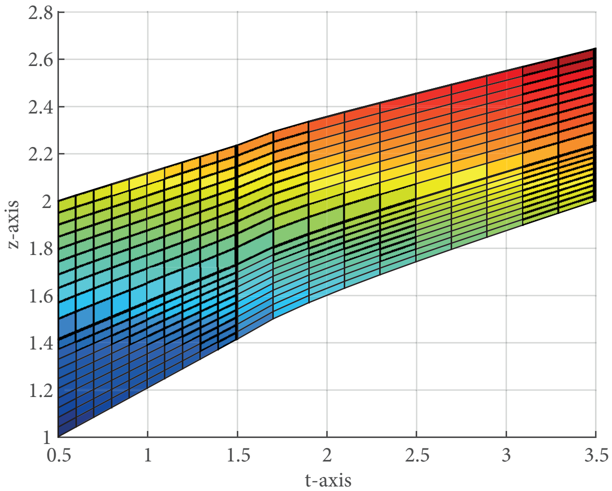


**Figure 5**. tz view of monotone surface generated from dataset in Table 3.

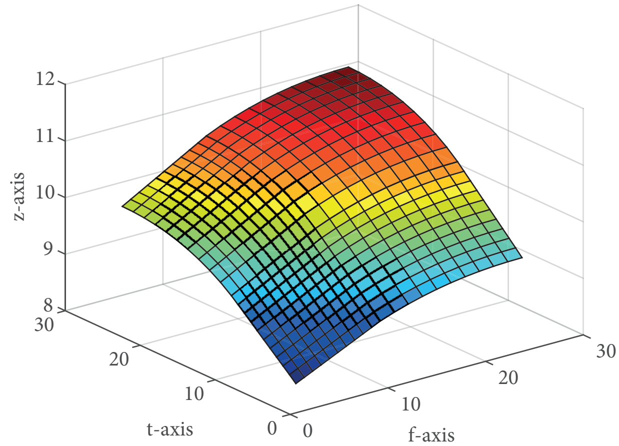


**Figure 6**. fz view of monotone surface generated from dataset in Table 3.

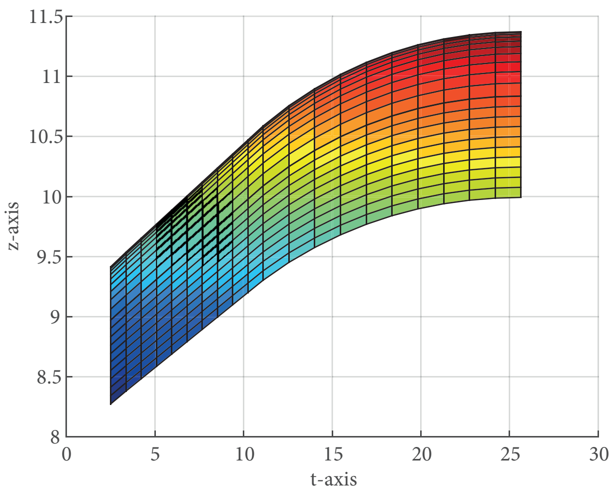


**Figure 7**. tz view of monotone surface generated from dataset in Table 4.

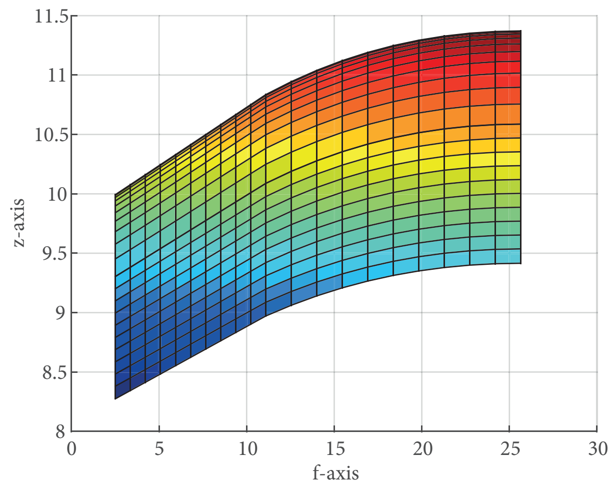


**Figure 8**. fz view of monotone surface generated from dataset in Table 4.

scientific phenomena. We then deduced certain constraints on both of these parameters to keep the shape of monotonic 2D data. To estimate the derivative values, the geometric mean approach has been used. The formulation of the monotonic $C^1$ continuous rational cubic spline function is extended to a monotone rational bicubic partially blended surface, where the data are guaranteed to be organized over a rectangular grid. Once again the parameters in the elucidation of the bicubic partially blended functions have been constrained for monotonicity. Eye-catching and appealing curves and surfaces are then acquired that can be used in systems that require such models.

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